E. Model contains log of time at risk as an offset

1. Fit component is added to every log rate

2. If you know something that rates might be proportional to, log of this could be added to the offset as well
   a. For ex, rate in unexposed population by age

F. Casting current models in the regression framework

1. One dimension:
   a. $\lambda_k = \exp(\alpha_k)$
   b. $\beta = (\alpha_0, \ldots, \alpha_{K-1})$, $z_k = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in position $k$.
   c. Model now has one parameter for every observation: saturated
   d. $L(\alpha) = \prod_{k=0}^{K-1} \exp([\omega_k + \alpha_k]X_k - \exp([\omega_k + \alpha_k]))/X_k!$
   e. $l(\alpha) = \sum_{k=0}^{K-1} \{[\alpha_k + \omega_k]X_k - \exp(\alpha_k + \omega_k) - \log(X_k!)]$
   f. $l^k(\alpha) = X_k - \exp(\alpha_k + \omega_k)$
   g. Maximizer satisfies $\hat{\alpha}_k = \log(X_k^\alpha) - \omega_k$
   h. For the submodel with all $\alpha$'s equal, $l(\alpha) = \alpha X_+ + \sum_{k=0}^{K-1} \omega_k X_k - \exp(\alpha) \sum_{k=0}^{K-1} \exp(\omega_k) - \sum_{k=0}^{K-1} \log(X_k!)$
i. \( l'(\alpha) = X_+ - \exp(\alpha) \sum_{k=0}^{K-1} \exp(\omega_k) \)

ii. \( \hat{\alpha} = \log(\frac{X_+}{\sum_{k=0}^{K-1} \exp(\omega_k)}) \).

iii. Profile score statistic is \( l^k(\alpha) = X_k - X_+ \exp(\omega_k)/\sum_{k=0}^{K-1} \exp(\omega_k) \)

i. After conditioning on \( X_+ \),

i. distribution is now multinomial with probabilities

\[
\pi_k = \frac{\exp(\omega_k + \alpha_k)}{\sum_{m=0}^{K-1} \exp(\omega_m + \alpha_m)}
\]

ii. Increasing or decreasing all of the \( \alpha_k \) by the same amount gives the same probabilities.

iii. Hence one can not identify all of the \( \alpha_k \).

iv. Pick one of these (ie., \( \alpha_0 = 0 \)), or set sum to zero (PROC CATMOD)

v. Likelihood is \( L(\alpha) = \left( \frac{X_+}{(X_0, \ldots, X_{K-1})} \right) \times \)

\[
\prod_{k=0}^{K-1} \exp([\omega_k + \alpha_k]X_k) \left[ \sum_{m=0}^{K-1} \exp(\omega_m + \alpha_m) \right]^{-X_+}.
\]

vi. Log likelihood is \( l(\alpha) = \log \left( \frac{X_+}{(X_0, \ldots, X_{K-1})} \right) + \)

\[
\sum_{k=0}^{K-1} [\omega_k + \alpha_k]X_k - X_+ \log \left[ \sum_{m=0}^{K-1} \exp(\omega_m + \alpha_m) \right].
\]

vii. \( l^k(\alpha) = X_k - X_+ \frac{\exp(\omega_k + \alpha_k)}{\sum_{m=0}^{K-1} \exp(\omega_m + \alpha_m)} \).
viii. Score is \( l^k(o) = X_k - X_+ \exp(\omega_k) \left/ \left[ \sum_{m=0}^{K-1} \exp(\omega_m) \right] \right. \).

A1: 6-6.1

G. Two Dimensions: \( \lambda_{k,j} = \exp(\mu + \alpha_k + \gamma_j + \sigma_{k,j}) \)

1. Clearly saturated: Could get any set of rates we want by setting \( \sigma_{k,j} \) equal to the associated rate.

2. Overparameterized: fix by
   a. setting \( \sigma_{k,j} = 0 \) if \( j = 0 \) or \( k = 0 \).
   b. Still over parameterized, since increasing \( \alpha \)'s by a constant, and decreasing the \( \gamma \)'s by the same amount, leaves model fits unchanged.

3. Odds ratio \( \theta_{k,j} = \frac{\exp(\mu + \alpha_k + \gamma_j + \sigma_{k,j}) \exp(\mu + \alpha_0 + \gamma_0 + \sigma_{00})}{\exp(\mu + \alpha_k + \gamma_0 + \sigma_{k0}) \exp(\mu + \alpha_0 + \gamma_j + \sigma_{0j})} = \exp(\sigma_{k,j}) \).

4. Model with no interactions: \( \sigma_{k,j} = 0 \) for all \( k, j \)
   a. Fix overparameterization by requiring \( \exp(\Sigma_j \gamma_j) = 1 \), \( \mu = 0 \)
   b. \( \frac{dl(\alpha, \gamma)}{d\alpha_k} = X_{k+} - \Sigma_j \exp(\alpha_k + \gamma_j) = X_{k+} - \exp(\alpha_k) \)
      i. Hence \( \hat{\alpha}_k = \log(X_{k+}) \)
   c. \( \frac{dl(\alpha, \gamma)}{d\gamma_j} = X_{+j} - \Sigma_k \exp(\alpha_k + \gamma_j) = X_{+j} - \exp(\gamma_j) \Sigma_k \exp(\alpha_k) \)
      i. Hence \( \hat{\gamma}_j = \log(X_{+j}/X_{++}) \)
d. Pearson goodness of fit statistic is test of $H_0: \sigma_{k,j} = 0 \forall j, k$, with $\alpha, \gamma$ arbitrary.

i. Only if data are arranged so that each cell in the full table is represented by one line in the data file.

5. Model with interactions:

a. Easiest constraint: $\mu = 0$, $\alpha_j = \gamma_k = 0$ for all $k, j$

b. $\hat{\sigma}_{k,j} = \log(X_{k,j})$

c. Under our earlier constraint that $\sigma_{k,j} = 0$ if $j = 0$, get

$$\hat{\sigma}_{k,j} = \log(X_{k,j}) - \log(X_{k0})$$

d. Under additional constraint that $\sigma_{k,j} = 0$ if $k = 0$, get

$$\hat{\sigma}_{k,j} = \log(X_{k,j}) - \log(X_{k0}) - \log(X_{0,j}) + \log(X_{00})$$: Earlier log odds ratio.

A: 7.1.4

H. Three Dimensions: $\lambda_{k,j}^i = \exp(\sigma_{k,j}^i)$

1. Again, $\hat{\sigma}_{k,j}^i = \log(X_{k,j}^i)$

2. Model forming null hypothesis for cmh test: $\lambda_{k,j}^i = \exp(\tau_{k,j} + \zeta_j^i + \xi_k^i)$

a. No closed form for MLE