VII. Sample Size Calculations

A. Preliminaries

1. We’ll do power for 1-sided tests
   a. Conceptually easier (as we shall see)
   b. Get power for 2-sided tests by doubling $\alpha$

B. Exactly:

1. Select smallest $C$ such that $P_0 [T \geq C] \leq \alpha$
2. Power is $P_A [T \geq C]$.

C. Approximately,

1. Suppose
   a. $H_0 : T \sim \mathcal{N}(\mu_0, \sigma_0^2)$
   b. $H_A : T \sim \mathcal{N}(\mu_A, \sigma_A^2)$
2. Critical value: $C'$
   a. Reject $H_0$ if $(T - \mu_0)/\sigma_0 \geq z_\alpha$
   b. $1 - \Phi(z_\alpha) = \alpha$
   c. Reject $H_0$ if $T \geq \mu_0 + \sigma_0 z_\alpha$
   d. $C' = \mu_0 + \sigma_0 z_\alpha$
3. Power is $P_A [T \geq C'] = \Phi((\mu_A - \mu_0 - \sigma_0 z_\alpha)/\sigma_A)$
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a. Special Case: $\sigma_0 = \sigma_A$, power is $\Phi((\mu_A - \mu_0)/\sigma_0 - z_\alpha)$

4. Sample size:

a. Assume that $\sigma_0 = \tau_0/\sqrt{n}$, $\sigma_A = \tau_A/\sqrt{n}$.

b. Require power $1 - \beta$

i. Typically, $.8 = 80\%$.

c. Then $-z_\beta = (\mu_0 + \sigma_0 z_\alpha - \mu_A)/\sigma_A$.

i. $(\sigma_A z_\beta + \sigma_0 z_\alpha) = \mu_A - \mu_0$

ii. $(\tau_A z_\beta + \tau_0 z_\alpha)/\sqrt{n} = \mu_A - \mu_0$

iii. $(\tau_A z_\beta + \tau_0 z_\alpha)/(\mu_A - \mu_0) = \sqrt{n}$

iv. $n = (\tau_A z_\beta + \tau_0 z_\alpha)^2/(\mu_A - \mu_0)^2$

v. When $\tau_A = \tau_0$, $n = \tau_0^2(z_\beta + z_\alpha)^2/(\mu_A - \mu_0)^2$

5. Example:

a. Null probability $.5$, alternative probability $.6$, one-sided test size $\alpha = .025$, power $1 - \beta = .8$

b. $z_\alpha = 1.96$, $z_\beta = 0.84$.

c. Need $X_+ = (\sqrt{.6 \times .4 z_{.8} - z_{.025}.5})^2 = 194$ individuals.

D. Often use variance stabilizing transformation for power

1. Suppose the parameter of interest is $\mu = E[T]$

2. Sometimes $\text{Var}[T]$ is a function of $\mu$. 
3. Look for transformation \( g(T) \) of test statistic \( T \) so that
\[
\text{Var} [g(T)] \text{ does not depend on } \mu
\]
4. \[
\text{Var} [g(T)] \approx \text{Var} [g(\mu) + g'(\mu)(T - \mu)] = g'(\mu)^2 \text{Var} [T].
\]
5. Find \( g \) so that \( g'(\mu) = 1/\text{Var} [T] \).
6. Ex., Poisson \( \text{Var} [T] = \mu \), so \( g'(\mu) = 1/\sqrt{\mu} \), \( g(\mu) = 2\sqrt{\mu} \).
7. Power is approximate
   a. Better approximation for Poisson uses fact that when
   \( X \sim \mathcal{P}(\mu) \) then \( \text{Var} [\sqrt{X}] \approx \mu \times (1/2\mu^{-1/2})^2 = \frac{1}{4} \).
   b. Better approximation for binomial uses fact that
   \[
   \arcsin(\sqrt{X_1/X_+}) \sim \mathcal{N}(\arcsin(\sqrt{Q_1/(Q_0 + \varsigma Q_1)}), 1/(4X_+))
   \]
   i. \( Q_+ \) is exponential of offset (or offset before you take log).
   ii. \( \frac{d}{dx} \arcsin(x) = 1/\sqrt{1 - x^2} \)
   iii. \( \frac{d}{dx} \arcsin(\sqrt{x}) = 1/\left(2\sqrt{x}\sqrt{1 - x}\right) \) See Figures 9 and 10.
5. \( \mu_0 = \arcsin(\sqrt{Q_1/(Q_0 + \varsigma Q_1)}), \mu_A = \arcsin(\varsigma \sqrt{Q_1/(Q_0 + \varsigma Q_1)}), \sigma_A = \sigma_0 = \sqrt{1/(4X_+)} \)
7. Exponential family models:
   a. Suppose that \( T \) has probabilities or mass function
Fig. 9: Sine Function

\[ \sin(x) \]

\[ \exp(t\tau - K(\tau) - c(t)) \]

i. Upper left corner of 2 \times 2 table fits, if \( \tau \) is log odds ratio

ii. If independent addends have this pattern, then so does sum.

b. Calculate \( K(\tau) = \log(E_0[\exp(\tau T)]) \)

c. Differentiating once,

\[
K'(\tau) = \frac{d}{d\tau} \frac{E[\exp(\tau T)]}{E_0[\exp(\tau T)]}
\]

\[
= E_0 \left[ \frac{d}{d\tau} \exp(\tau T) \right] / E_0[\exp(\tau T)]
\]

\[
= E_0 [T \exp(\tau T)] / E_0[\exp(\tau T)] = E_\tau [T]
\]

d. Differentiating again,
Fig. 10: Variance Stabilizing Transformation

\[
\text{arcsin}(\sqrt{p})
\]

\[
\begin{align*}
K''(\tau) &= \frac{d}{d\tau} \left[ \frac{E_0 [T \exp(\tau T)]}{E_0 [\exp(\tau T)]]} \right] \\
&= \frac{d}{d\tau} \frac{E_0 [T \exp(\tau T)]}{E_0 [\exp(\tau T)]} - \frac{d}{d\tau} \frac{E_0 [\exp(\tau T)] E_0 [T \exp(\tau T)]}{E_0 [\exp(\tau T)]^2} \\
&= \frac{E_0 [T^2 \exp(\tau T)]}{E_0 [\exp(\tau T)]} - \frac{E_0 [T \exp(\tau T)] E_0 [T \exp(\tau T)]}{E_0 [\exp(\tau T)]^2} \\
&= E_\tau [T^2] - E_\tau [T]^2 = \text{Var}_\tau [T]
\end{align*}
\]

e. So \( \frac{d}{d\tau} E_\tau [T] = \text{Var}_\tau [T] \)

f. If \( H_0 : \tau = 0 \), \( H_A : \tau = \theta \), then for small \( \theta \), \( \sigma_0^2 \approx \sigma_A^2 \), \\
\( \mu_A - \mu_0 \approx \sigma_0^2 \theta \)
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g. Power is $\Phi(\sigma_0 \theta - z_\alpha)$

9. Mantel–Haenszel example:
   a. $T =$ sum of upper right corners
   b. $\sigma_0 = \sqrt{\sum_i \frac{X_{1i}^i X_{0i}^i + X_{1i}^i X_{+0}^i}{X_{++}^i X_{++}^i (X_{++}^i - 1)}}$
   c. $X_{0+}$, $X_{1+}$, and $X_{++}$ fixed in advance.
   d. $X_{+j}$ should be replaced by $X_{++} \pi_{+j}$

10. Mantel-Haentzel Example: Henhouse data set
   a. Six labs, and expect 9 control and 9 treatment chicks per lab
      i. So $X_{0+}^i = X_{1+}^i = 9$
   b. Expect null proportions of abnormalities to be $1/9$ to $6/9$.
      i. So null column totals are $X_{+0}^i = (2, 4, 6, 8, 10, 12)$, $X_{+1}^i = (16, 14, 12, 10, 8, 6)$.
   c. Null variance is $9 \times 9 \times (2 \times 16 + 4 \times 14 + 6 \times 12 + 8 \times 10 + \ldots)/(18 \times 18 \times 17) = 5.76$, and SE is $2.40$
   d. Power for detecting log odds ratio of .5 is $\Phi(2.40 \times .5 - 1.96) = .224$.
   e. For 80% power, need log odds ratio satisfying $2.27 \tau - 1.96 = .84$ or $\tau = (0.84 + 1.96)/2.27$.
   f. If you want 80% power with log odds ratio .5, need
$\sigma_0 \times .5 - 1.96 = .84$, or $\sigma_0 = (0.84 + 1.96)/.5 = 5.6$.

Since $\sigma_0$ is approximately proportional to the square root of the number of chicks per group per lab, you need $9 \times (5.6/2.27)^2$ chicks per group in each lab.