VII. Sample Size Calculations

A. Preliminaries
1. We’ll do power for 1-sided tests
   a. Conceptually easier (as we shall see)
   b. Get power for 2-sided tests by doubling \( \alpha \)

2. Critical value: \( C \)
   a. Reject \( H_0 \) if \( (T - \mu_0)/\sigma_0 \geq z_\alpha \)
   b. Reject \( H_0 \) if \( T \geq \mu_0 + \sigma_0 z_\alpha \)
   c. \( C = \mu_0 + \sigma_0 z_\alpha \)

3. Power is \( P_T[T \geq C] = \Phi((\mu_A - \mu_0 - \sigma_0 z_\alpha)/\sigma_A) \)
   a. Special Case: \( \sigma_0 = \sigma_A \), power is
   \[ \Phi((\mu_A - \mu_0)/\sigma_A - z_\alpha) \]

4. Sample size:
   a. Assume that \( \sigma_0 = \tau_0/\sqrt{n} \), \( \sigma_A = \tau_A/\sqrt{n} \).
   b. Require power \( 1 - \beta \)
      i. Typically, \( \beta = .8 \)
   c. Then \( -z_\beta = (\mu_0 + \sigma_0 z_\alpha - \mu_A)/\sigma_A \).
      i. \( (\sigma_A z_\beta + \tau_0 z_\alpha)/\sqrt{n} = \mu_A - \mu_0 \)
      ii. \( (\tau_A z_\beta + \tau_0 z_\alpha)/\sqrt{n} = \mu_A - \mu_0 \)
      iii. \( (\tau_A z_\beta + \tau_0 z_\alpha)/(\mu_A - \mu_0) = \sqrt{n} \)

8. Exponential family models:
   a. Suppose that \( T \) has probabilities or mass function
      \[ \exp(t \tau - K(\tau) - c(t)) \]
      i. Upper left corner of \( 2 \times 2 \) table fits, if \( \tau \) is log odds ratio
      ii. If independent addends have this pattern, then so does sum.
   b. Calculate \( K(\tau) = \log(E_0[\exp(\tau T)]) \)
   c. Differentiating once,
      \[ K'(\tau) = \frac{d}{d\tau} \log(E_0[\exp(\tau T)]) \]
      \[ = \frac{d}{d\tau} \log(E_0[\exp(\tau T)]) + \log(E_0[\exp(\tau T)]) \]
      \[ = E_T[\exp(\tau T)]/E_0[\exp(\tau T)] \]
      \[ = E_T[T\exp(\tau T)]/E_0[\exp(\tau T)] = E_T[T] \]
   d. Differentiating again,
\[K''(\tau) = \frac{d}{d\tau} \left[ \frac{E_0 [T \exp(\tau T)]}{E_0 [\exp(\tau T)]} \right] \]
\[= \frac{d}{d\tau} E_0 [T \exp(\tau T)] - \frac{d}{d\tau} E_0 [\exp(\tau T)] E_0 [T \exp(\tau T)] \frac{E_0 [\exp(\tau T)]^2}{E_0 [\exp(\tau T)]} \]
\[= E_0 [T] - E_0 [\tau T]^2 = \text{Var}_\tau [T] \]

e. So \( \frac{d}{d\tau} E_\tau [T] = \text{Var}_\tau [T] \)
f. If \( H_0: \tau = 0, H_A: \tau = \theta \), then for small \( \theta \),
\[\sigma_0^2 = \sigma_A^2, \mu_A - \mu_0 = \sigma_0^2 \theta \]
g. Power is \( \Phi(\sigma_0 \theta - z_\alpha) \)

9. Mantel–Haenszel example:
   a. \( T = \) sum of upper right corners
   b. \( \sigma_0 = \sqrt{\sum_i \frac{X_{i+}^i X_{i+}^i X_{i+} X_{i+}^i}{X_{i+} X_{i+} X_{i+} - 1}} \)
   c. \( X_{0+}, X_{1+}, \) and \( X_{++} \) fixed in advance.
   d. \( X_{+j} \) should be replaced by \( X_{++} \pi_{+j} \)

10. Mantel-Haentzel Example: Henhouse data set
   a. Six labs, and expect 9 control and 9 treatment chicks per lab
   i. So \( X_{0+}^i = X_{i+}^i = 9 \)
   b. Expect null proportions of abnormalities to be \( 1/9 \) to \( 6/9 \),
   i. So null column totals are \( X_{1+0}^i = (2, 4, 6, 8, 10, 12), \)
   \( X_{1+1}^i = (16, 14, 12, 10, 8, 6) \).