Problem 1. Let $Y_1, \ldots, Y_n$ be i.i.d. random samples from the family of distributions with pdf $\{f_{\theta}(y) : \theta \in \Theta\}$. Find the maximum likelihood estimator (MLE) of $\theta$ in each of the following cases:

1. $f_{\theta}(y) = \frac{\theta}{y^2}, 0 < \theta \leq y < \infty$.
2. $f_{\theta}(y) = \theta y^{\theta - 1}, 0 < y < 1, 0 < \theta < \infty$.
3. $f_{\theta}(y) = \theta^y(1 - \theta)^{y}, 0 \leq y \leq 1, 1/2 \leq \theta \leq 3/4$.

Problem 2. Prove the following:

1. $o_p(1) + O_p(1) = O_p(1)$.
2. $o_p(1) \cdot O_p(1) = o_p(1)$.

Problem 3. Let $Y_1, \ldots, Y_n$ be i.i.d. samples from the class of uniform distributions on $[0, \theta]$. Show that the maximum likelihood estimator fails to be asymptotically normal, and derive its explicit limit distribution.

Problem 4. Let $Y_1, \ldots, Y_n$ be i.i.d. samples from an unknown distribution. Let $S^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$ be the sample variance. Suppose $EY_1^4 < \infty$. Find the limit distribution for

$$\sqrt{n}\left(\bar{Y} - \frac{EY_1}{\text{Var}(Y_1)}\right).$$

Problem 5. Suppose that $(X_i, Y_i) \sim_{\text{i.i.d.}} N((\mu_i, \mu_i), \sigma^2 I)$ for $i = 1, \ldots, n$, where $\{\mu_i\}_{i=1}^{n}, \sigma^2$ are unknown. Find the MLE $\hat{\sigma}^2_n$ for $\sigma^2$, and show that $\hat{\sigma}^2$ is inconsistent for estimating $\sigma^2$.

Problem 6. Let $Y_1, \ldots, Y_n$ be i.i.d. random samples from density $f_{\theta}(y)$ where $\theta_0 = (\eta_0, \lambda_0) \in \mathbb{R}^2$. Let $I_0 = I(\theta_0)$ be the Fisher information, and $\hat{\theta}_n = (\hat{\eta}_n, \hat{\lambda}_n)$ be the maximum likelihood estimator.

1. Show that $\sqrt{n}(\hat{\eta}_n - \eta_0) \rightarrow_d N(0, \tau^2)$, and given an explicit formula for $\tau$ using $I_0$.
2. Let $\tilde{\eta}_n$ be the maximum likelihood estimator with $\lambda_0$ known. Show that $\sqrt{n}(\tilde{\eta}_n - \eta_0) \rightarrow_d N(0, \nu^2)$, and that $\nu^2 \leq \tau^2$. 
Problem 7. Let $S_1$ and $S_2$ be independent with $S_j \sim (c_j + \theta)\chi^2(n_j)$ for $j = 1, 2$ with $\theta \geq 0$ an unknown parameter and with $0 < c_1 \leq c_2$ known constants. A statistician wishes to estimate $\theta$, having observed $S_1$ and $S_2$.

(1) For this part and the next, assume that $c_1 = c_2$. Construct the MLE of $\theta$ and the UMVUE, keeping in mind that $\theta$ is constrained to be nonnegative (so the domain of the likelihood function is $[0, 1]$). Are these necessarily the same? Does the score function necessarily have a first derivative? Discuss whether the MLE or the UMVUE is better.

(2) For the MLE, determine the Fisher information evaluated at the MLE.

(3) Now and for the rest of the problem, suppose $c_1 < c_2$. Show that neither $(S_1, S_2)$ nor any linear combination $a_1S_1 + a_2S_2$ is a complete sufficient statistic.

(4) Evaluate the score function and the Fisher information $\ell(\theta)$. Assuming that the likelihood function has a stationary point at some $\theta > 0$, show how one can solve for the root by Newton’s method, i.e., give the updating equation for how to get the next estimate as a function of the current estimate. Now show how one solves for the root by Fishers method of scoring. Show that root finding by Fishers method of scoring is equivalent to iteratively calculating the best weighted average of the separate unbiased estimates of $\theta$, one based on $S_1$ and the other based on $S_2$. 