11) \( f(y|\theta) = \begin{cases} \theta y^{\theta - 1}, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \)

Let \( X = -\log y \), then \( y = e^{-X} \).

\[
f_X(x) = \int_{e^{-x}} f_y(e^{-x}) e^{-x} \, dx
= \theta (e^{-x})^{\theta - 1} | e^{-x} |
= \theta e^{-\theta x}
\]

\( X \sim f_X(x) \sim \text{Gamma}(1, \theta) \).

\( T = -\sum \log y_i = -\sum X_i \sim \text{Gamma}(n, \theta) \).

(2) \[
\frac{f(t|\theta)}{f(t|\theta_1)} = \left( \frac{\theta_1}{\theta} \right)^n t^{n-1} e^{-t(\theta_1 - \theta)}
\]

Since \( \theta > \theta_1 \) and \( \theta_2 - \theta > 0 \), it's non-increasing likelihood ratio.

(3) \[
\log \left( \frac{L(\theta_2)}{L(\theta_1)} \right) = n \log \theta + (\theta_1 - \theta_1) \sum \log y_i
= n \log \theta + (\theta_1 - \theta) \sum \frac{1}{e} \log y_i
\]

\[
\frac{d \log L(\theta)}{d \theta} = \frac{n}{\theta} + \frac{1}{\theta} \sum \log y_i \geq 0
\]

\[
\theta_{M_2} = -\frac{n}{\sum \log y_i} = \frac{1}{\bar{y}} \quad \text{for} \quad 0 < y < 1
\]

\[
\frac{d \log L(\theta)}{d \theta} = -\frac{n \theta}{\sum \log y_i} \quad \text{so} \quad \theta_{M_2} = \frac{1}{\bar{y}} \text{ is maximum likelihood estimator}.
\]

(4) \( H_0: \theta = \theta_0 \) vs \( H_1: \theta \neq \theta_0 \)

\[
\chi^2 = \frac{\sum (\log y_i - \log \hat{\theta}_{M_2})^2}{\sum (\log y_i - \log \hat{\theta}_{M_2})^2}
\]

\[
\log \chi^2 = \log L(\theta_0|y) - \log L(\hat{\theta}_{M_2}|y)
= n \log \theta_0 - (\theta_0 - \hat{\theta}_{M_2}) \sum \log y_i
= n \log \theta_0 - (\theta_0 - \hat{\theta}_{M_2}) \sum \frac{1}{e} \log y_i
= n \log \frac{\theta_0}{\hat{\theta}_{M_2}} - \frac{n \theta_0}{\hat{\theta}_{M_2}} + n
= n \left( \log \frac{\theta_0}{\hat{\theta}_{M_2}} + \frac{n \theta_0}{\hat{\theta}_{M_2}} + 1 \right) \leq \log k
\]}
Let \( \phi(\frac{\theta_0}{\bar{\theta}}) = \log \frac{\theta_0}{\bar{\theta}} - \frac{\theta_0}{\bar{\theta}} + 1 \)

\( \phi(x) = \log x - x + 1 \)

\( \phi'(x) = \frac{1}{x} - 1 \)

Since \( \theta > 0 \) and \( \frac{\theta_0}{\bar{\theta}} > 0 \), \( x > 0 \).

\( \phi(x) = \frac{1}{x} - 1 \)

> 0 if \( x < 1 \)

< 0 if \( x > 1 \)

\( \phi(1) = 0 - 1 + 1 = 0 \)

\( \phi(x) \) is increasing when \( x < 1 \)

\( \phi(x) \) is decreasing when \( x > 1 \)

Rejection region \( \log \lambda(x) < \log \lambda \)

\( \phi(\frac{\theta_0}{\bar{\theta}}) < \frac{\log \lambda}{n} \)

Reject if \( \frac{\theta_0}{\bar{\theta}} < a \) or \( \frac{\theta_0}{\bar{\theta}} > b \)

That is \( \frac{\theta_0}{\bar{\theta}} < \hat{\theta} < \frac{\theta_0}{b} \)

where \( \phi(a) = \phi(b) = \frac{\log \lambda}{n} \)
For $\theta > \theta_0$,

$$\frac{f(T|\theta_1)}{f(T|\theta_0)} = \left(\frac{\theta_0}{\theta_1}\right)^{\theta_0-\theta_1} e^{-(\theta_1 T - \theta_0 T)} = \left(\frac{\theta_0}{\theta_1}\right)^{\theta_0-\theta_1} e^{-T(\theta_1-\theta_0)}$$

is a decreasing function of $T$.

Also, $T$ is sufficient statistic.

Hence, it has MLE. By Koopman-Rubin,

the UMP for $H_0: \theta = \theta_0$ vs $H_1: \theta < \theta_0$ has rejection region of the form

$$\{T > t_0\}$$

where

$$\Gamma = \sup P(T > t_0 | \theta = \theta_0)$$

Since the pdf of $T$ is statistically increasing,

$$\sup P(T > t_0 | \theta < \theta_0) = P(T > t_0 | \theta = \theta_0)$$

$$= P(2\theta_0 T > t_0 | \theta = \theta_0)$$

Since $T \sim \text{Gamma}(n, 1)$, hence under $\theta = \theta_0$, $2\theta_0 T \sim \text{Gamma}(n, 2)$, i.e. $2\theta_0 T \sim \chi^2(2n)$.

Hence $2\theta_0 T \sim \chi^2(2n)$ implies $t_0 = \frac{\chi^2_{2n}(\alpha)}{2\theta_0}$.

Thus, the UMP (one-tailed) test has rejection region $\{T > \frac{\chi^2_{2n}(\alpha)}{2\theta_0}\}$.
(b) \( T \sim \text{Gamma}(n, \frac{1}{\theta}) \), thus \( T \sim \text{Gamma}(n, 1) \) whose distribution does not depend on \( \theta \). Thus, \( T \) is a pivot.

The confidence interval takes the form \( \{ \theta : a \leq T \leq b \} \), and

\[
\int_a^b f_{\theta}(x)dx = 1 - \alpha. \quad \text{Note that Gamma}(n, 1) \) is unimodal. Thus, by setting \( f_{\theta}(a) = f_{\theta}(b) \), we can get the shortest interval.
\]

\[
\frac{a^{n-1}e^{-a}}{\Gamma(n)} = \frac{b^{n-1}e^{-b}}{\Gamma(n)} \implies -a + (n-1)\ln a = -b + (n-1)\ln b \quad \text{(1)}
\]

\[
\int_a^b \frac{x^{n-1}e^{-x}}{\Gamma(n)} dx = 1 - \alpha \quad \text{(2)}
\]

Solving \( 1 \) & \( 2 \) we can get \( a \) and \( b \), then the \( (1 - \alpha) \) CI is

\[
C(T) = \{ \theta : \frac{a}{1} \leq \theta \leq \frac{b}{1} \}.
\]
2) \[ L(\theta|y) = \frac{2^n}{\theta^n} \prod_{i=1}^{n} I(y_i \leq \theta) I(y_n > \theta) \]

We can see \( L(\theta|y) \) decreases as \( \theta \) increases. Thus, the maximum of \( L(\theta|y) \) is achieved by the smallest value of \( \theta \). Hence \( \hat{\theta} = y(n) \).

2) \[
\hat{\theta}_0 = \begin{cases} \theta_0 & y(n) \leq \theta_0 \\
\hat{\theta} = y(n) & y(n) > \theta_0 \end{cases}
\]

Hence, \( \Delta(y) = \frac{L(\hat{\theta}|y)}{L(\theta_0|y)} = \left( \frac{y(n)}{\theta_0} \right)^{-n} \) if \( y(n) \leq \theta_0 \).

Thus, the LRT has rejection region \[ \{ y|y : \left( \frac{y(n)}{\theta_0} \right)^{-n} < c \text{ and } y(n) \leq \theta_0 \} \]

3) First find the density of \( y(n) \)

\[ P(y(n) \leq y) = \left[ P(X \leq y) \right]^n = \left[ \int_{-\infty}^{y} \frac{2}{\theta^2} \exp\left(-\frac{t}{\theta}\right) \, dt \right]^n = \left( \frac{y}{\theta} \right)^n \quad \text{for } y \leq \theta. \]

\[ f_{y(n)}(y) = \frac{2ny^{2n-1}}{\theta^{2n}} \quad \text{for } 0 \leq y \leq \theta. \]

Let \( \theta_1 > \theta_2 \):

\[ \frac{f_{y(n)}(y|\theta_1)}{f_{y(n)}(y|\theta_2)} = \frac{\theta_2^{2n} 1(y \leq \theta_1)}{\theta_1^{2n} 1(y \leq \theta_2)} = \begin{cases} \left( \frac{\theta_2}{\theta_1} \right)^{-n} & y < \theta_2 < \theta_1 \\
\infty & \theta_2 < y < \theta_1 \\
\text{not defined} & y > \theta_1 \end{cases} \]

Thus, it has MLE. Also, \( y(n) \) is sufficient statistic.

The rejection region has the form \[ \{ y(n) < t \} \]
\[ a = \sup_{\theta \geq \theta_0} P_\theta(Y_n < t) = \sup_{\theta \geq \theta_0} \int_0^t \frac{2n \gamma^{2n-1}}{\theta^{2n}} \, dy = \sup_{\theta \geq \theta_0} \left[ -\left(\frac{t}{\theta}\right)^n \right] = \left(\frac{t}{\theta_0}\right)^n \]

Thus, \[ t = a^{1/n} \theta_0 \]

The UMP test has rejection region \( \{ Y_n < a^{1/n} \theta_0 \} \)

(4) The acceptance region of (3) is \( A(\theta_0) = \{ Y_n > a^{1/n} \theta_0 \} \). Note that \( P_{\theta_0}(Y_n > \theta_0) = 0 \).

Thus, the acceptance region is equivalent to \( A(\theta_0) = \{ a^{1/n} \theta_0 < Y_n < \theta_0 \} \).

Thus, the confidence interval is \( C(Y_n) = \{ Y_n \leq \theta_0 \leq Y(n) a^{1/n} \} \).

(5) We have shown \( f(Y_n; \theta) \) has MLE. Thus, the rejection region for this test takes the form: \( \{ Y_n > t \} \), and

\[ a = \sup_{\theta \geq \theta_0} P_\theta(Y_n > t) = \sup_{\theta \geq \theta_0} \int_0^t \frac{2n \gamma^{2n-1}}{\theta^{2n}} \, dy = \sup_{\theta \geq \theta_0} \left[ 1 - \left(\frac{t}{\theta}\right)^n \right] = 1 - \left(\frac{t}{\theta_0}\right)^n \]

Thus, \[ t = (1-a)^{1/n} \theta_0 \]

Hence, the rejection region for UMP level \( a \) test is \( \{ Y_n : Y_n > (1-a)^{1/n} \theta_0 \} \).
Claim: the UMP level \( \alpha \) test for \( H_0: \theta = \theta_0 \) vs. \( H_1: \theta = \theta_1 \) has the rejection region \( R = \{ Y_{1n}: Y_{1n} < \alpha^{1/n} \theta_0 \text{ or } Y_{1n} > \theta_0 \} \). Denote its power function by \( p^*(\theta) \).

For \( \theta_1 < \theta_0 \). We have shown the UMP test of \( H_0: \theta = \theta_0 \) vs. \( H_1: \theta = \theta_1 \) should have a rejection region \( R = \{ Y_{1n}: Y_{1n} < \alpha^{1/n} \theta_0 \} \).

The power of the test at \( \theta_1 \) is \( \beta(\theta_1) = P \left( Y_{1n} < \alpha^{1/n} \theta_0 \mid \theta = \theta_1 \right) \).

And \( p^*(\theta_1) = P \left( Y_{1n} < \alpha^{1/n} \theta_0 \text{ or } Y_{1n} > \theta_0 \mid \theta = \theta_1 \right) = 0 \) since \( P \left( Y_{1n} > \theta_0 \mid \theta = \theta_1 \right) = 0 \).

Hence, \( p^*(\theta_1) = P \left( Y_{1n} < \alpha^{1/n} \theta_0 \right) = \beta(\theta_1) \).

For \( \theta_1 > \theta_0 \). We have shown the UMP test of \( H_0: \theta = \theta_0 \) vs. \( H_1: \theta = \theta_1 \) should have a rejection region \( R = \{ Y_{1n}: Y_{1n} > (1-\alpha)^{1/n} \theta_0 \} \).

The power at \( \theta_1 \) is \( \beta(\theta_1) = P \left( Y_{1n} > (1-\alpha)^{1/n} \theta_0 \mid \theta = \theta_1 \right) = \int_{(1-\alpha)^{1/n} \theta_0}^{\infty} f_{Y_{1n}}(y) \, dy 

= 1 - (1-\alpha) \left( \frac{\theta_0}{\theta_1} \right)^{2n} \).

And \( p^*(\theta_1) = P \left( Y_{1n} > \theta_0 \text{ or } Y_{1n} < \alpha^{1/n} \theta_0 \mid \theta = \theta_1 \right) = \int_0^{\alpha^{1/n} \theta_0} f_{Y_{1n}}(y) \, dy + \int_{(1-\alpha)^{1/n} \theta_0}^{\infty} f_{Y_{1n}}(y) \, dy = \alpha \cdot \left( \frac{\theta_0}{\theta} \right)^{2n} + 1 - \left( \frac{\theta_0}{\theta} \right)^{2n} = 1 - (1-\alpha) \left( \frac{\theta_0}{\theta} \right)^{2n} = p^*(\theta_1) \).

Hence, the test with rejection region \( R \) has the same power as the UMP level \( \alpha \) test for \( \theta_1 \). Thus, by the optimality of Neyman-Pearson lemma, the test with rejection region \( R \) is UMP level \( \alpha \) test for \( H_0: \theta = \theta_0 \) vs. \( H_1: \theta = \theta_1 \).
Consider

\[ H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1 \ (\theta_0 < \theta_1) \]

Then

| \( x(n) \leq \theta_1 \) | \( f(x | \theta_0) \) | \( f(x | \theta_1) \) |
|------------------------|----------------|----------------|
| \( 0 < x(n) \leq \theta_0 \) | \( \left( \frac{2}{\theta_0^n} \right) x_1 \cdots x_n \) | \( \left( \frac{2}{\theta_1^n} \right) x_1 \cdots x_n \) |
| \( x(n) > \theta_0 \) | \( 0 \) | \( 0 \) |

Note that the rejection region \( R^* = \{ x(n) \leq \frac{1}{n} \theta_0 \} \) has size \( \alpha \).

Case I: If \( x(n) \leq \theta_0 \), then \( R^* \) satisfies

\[
\text{if } f(x | \theta_1) > k f(x | \theta_0) \quad \text{then } x \in R^* \]

\[
\text{if } f(x | \theta_1) < k f(x | \theta_0) \quad \text{then } x \in R' \]

with the choice \( k = 0 \). The above condition simply says

\[
\text{if } x(n) \leq \theta_0 \quad \text{then } x \in R^* \]

By necessity of Neyman Lemma, any UMP level \( \alpha \) test must satisfy \( \text{if } x \in R \) or equivalently \( \text{if } x \in R' \). Therefore, a UMP level \( \alpha \) rejection region \( R \) must be of the form

\[
R = \{ x(n) \leq \theta_1 \} \cup A \quad \text{such that } P_{\theta_0}(R) \leq \alpha.
\]

for an arbitrary set \( A \).
It is easy to see that all such UMP level \( \alpha \) tests have

\[ \text{power} = 1, \text{ i.e. } P_{0_1}(R) = 1. \]

\textbf{Case II:}

If \( x_2 = 0_2 < 0_1 \), then \( R^* \) satisfies \( \bigcirc \)

with the choice \( k = \left( \frac{0_2}{0_1} \right)^{2\alpha} \). The condition \( \bigcirc \) simply requires

\[ \text{if } x_n > 0_1 \text{ then } x \in R^*. \]

By necessity of NP lemma, any UMP level \( \alpha \) test must

\[ \text{satisfy } \bigcirc \text{ with } k = \left( \frac{0_2}{0_1} \right)^{2\alpha} \text{ or equivalently } \bigcirc \bigcirc. \]

Therefore, a UMP level \( \alpha \) rejection region \( R \) must be of the form

\[ R = \left\{ x_n \leq 0_1 \right\} \cap A \]

for an arbitrary set \( A \) such that \( P_{0_2}(R) = \alpha. \)

The power of such UMP level \( \alpha \) tests is

\[ P_{0_1}(R) = \int_{R} \left( \frac{z}{\theta_1} \right)^{n} x_1 - x_n \, dx_1 - dx_n \]

\[ = \left( \frac{0_2}{0_1} \right)^{2\alpha} \int_{R} \left( \frac{z}{\theta_0} \right)^{n} x_1 - x_n \, dx_1 - dx_n \]

\[ = \left( \frac{0_2}{0_1} \right)^{2\alpha} \, \alpha. \]