6.6 The joint pdf is given by
\[ f(x_1, \ldots, x_n|\alpha, \beta) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta} = \left( \frac{1}{\Gamma(\alpha)\beta^\alpha} \right)^n \left( \prod_{i=1}^{n} x_i \right)^{\alpha-1} e^{-\Sigma x_i/\beta}. \]

By the Factorization Theorem, \( \prod_{i=1}^{n} X_i \) is sufficient for \((\alpha, \beta)\).

6.7 Let \( x_{(1)} = \min\{x_1, \ldots, x_n\}, \quad x_{(n)} = \max\{x_1, \ldots, x_n\} \), \( y_{(1)} = \min\{y_1, \ldots, y_n\} \) and \( y_{(n)} = \max\{y_1, \ldots, y_n\} \). Then the joint pdf is
\[
\begin{align*}
    f(x|y|\theta) & = \prod_{i=1}^{n} \frac{1}{(\theta_3 - \theta_1)(\theta_4 - \theta_2)} \mathcal{I}(\theta_1, \theta_3)(x_i) \mathcal{I}(\theta_2, \theta_4)(y_i) \\
    & = \left( \frac{1}{(\theta_3 - \theta_1)(\theta_4 - \theta_2)} \right)^n \mathcal{I}(\theta_1, \infty)(x_{(1)}) \mathcal{I}(\infty, \theta_3)(x_{(n)}) \mathcal{I}(\theta_2, \infty)(y_{(1)}) \mathcal{I}(\infty, \theta_4)(y_{(n)}) \cdot \frac{1}{\eta(x)}.
\end{align*}
\]

By the Factorization Theorem, \( (X_{(1)}, X_{(n)}, Y_{(1)}, Y_{(n)}) \) is sufficient for \((\theta_1, \theta_2, \theta_3, \theta_4)\).

6.9 Use Theorem 6.2.13.

a. \[
\frac{f(x|\theta)}{f(y|\theta)} = \frac{(2\pi)^{-n/2} e^{-\Sigma x_i - \Sigma y_i}/2}{(2\pi)^{-n/2} e^{-\Sigma y_i - y_i^2}/2} = \exp \left\{ -\frac{1}{2} \left( \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 + 2\theta n(y - \bar{x}) \right) \right\}.
\]

This is constant as a function of \(\theta\) if and only if \(\bar{y} = \bar{x}\); therefore \(X\) is a minimal sufficient statistic for \(\theta\).

b. Note, for \(X \sim \text{location exponential}(\theta)\), the range depends on the parameter. Now
\[
\begin{align*}
    f(x|\theta) & = \prod_{i=1}^{n} \left( e^{-(x_i - \theta)} \mathcal{I}(\theta, \infty)(x_i) \right) \\
    f(y|\theta) & = \prod_{i=1}^{n} \left( e^{-(y_i - \theta)} \mathcal{I}(\theta, \infty)(y_i) \right) \\
    & = e^{n\theta} e^{-\Sigma x_i} \prod_{i=1}^{n} \mathcal{I}(\theta, \infty)(x_i) e^{n\theta} e^{-\Sigma y_i} \prod_{i=1}^{n} \mathcal{I}(\theta, \infty)(y_i) \\
    & = e^{-\Sigma x_i} \mathcal{I}(\theta, \infty)(\min x_i) e^{-\Sigma y_i} \mathcal{I}(\theta, \infty)(\min y_i).
\end{align*}
\]

To make the ratio independent of \(\theta\) we need the ratio of indicator functions independent of \(\theta\). This will be the case if and only if \(\min\{x_1, \ldots, x_n\} = \min\{y_1, \ldots, y_n\}\). So \(T(X) = \min\{X_1, \ldots, X_n\}\) is a minimal sufficient statistic.

c. \[
\begin{align*}
    \frac{f(x|\theta)}{f(y|\theta)} & = \frac{e^{-\Sigma (x_i - \theta)}}{\prod_{i=1}^{n} \left( 1 + e^{-(x_i - \theta)} \right)^2} \frac{\prod_{i=1}^{n} \left( 1 + e^{-(y_i - \theta)} \right)^2}{e^{-\Sigma (y_i - \theta)}} \\
    & = e^{-\Sigma (y_i - x_i)} \left( \frac{\prod_{i=1}^{n} \left( 1 + e^{-(y_i - \theta)} \right)}{\prod_{i=1}^{n} \left( 1 + e^{-(x_i - \theta)} \right)} \right)^2.
\end{align*}
\]

This is constant as a function of \(\theta\) if and only if \(x\) and \(y\) have the same order statistics. Therefore, the order statistics are minimal sufficient for \(\theta\).

d. This is a difficult problem. The order statistics are a minimal sufficient statistic.
e. Fix sample points x and y. Define $A(\theta) = \{i : x_i \leq \theta\}$, $B(\theta) = \{i : y_i \leq \theta\}$, $a(\theta) =$ the number of elements in $A(\theta)$ and $b(\theta) =$ the number of elements in $B(\theta)$. Then the function $f(x|\theta)/f(y|\theta)$ depends on $\theta$ only through the function

$$\sum_{i=1}^{n} |x_i - \theta| - \sum_{i=1}^{n} |y_i - \theta| = \sum_{i \in A(\theta)} (\theta - x_i) + \sum_{i \in A(\theta)} (x_i - \theta) - \sum_{i \in B(\theta)} (\theta - y_i) - \sum_{i \in B(\theta)} (y_i - \theta) = (a(\theta) - [n - a(\theta)]) - b(\theta) + [n - b(\theta)]\theta + \left( - \sum_{i \in A(\theta)} x_i + \sum_{i \in A(\theta)} x_i + \sum_{i \in B(\theta)} y_i - \sum_{i \in B(\theta)} y_i \right).$$

Consider an interval of $\theta$s that does not contain any $x_i$ or $y_i$. The second term is constant on such an interval. The first term will be constant on the interval if and only if $a(\theta) = b(\theta)$. This will be true for all such intervals if and only if the order statistics for $x$ are the same as the order statistics for $y$. Therefore, the order statistics are a minimal sufficient statistic.

---

6.10 To prove $T(X) = (X(1), X(n))$ is not complete, we want to find $g(T(X))$ such that $E[g(T(X))] = 0$ for all $\theta$, but $g(T(X)) \neq 0$. A natural candidate is $R = X(n) - X(1)$, the range of $X$, because by Example 6.2.17 its distribution does not depend on $\theta$. From Example 6.2.17, $R$ ~ beta($n-1, 2$). Thus $E(R) (n-1)/(n+1)$ does not depend on $\theta$, and $E(R - E(R)) = 0$ for all $\theta$. Thus $g(X(n), X(1)) = X(n) - X(1) - (n-1)/(n+1) = R - E(R)$ is a nonzero function whose expected value is always 0. So, $(X(1), X(n))$ is not complete. This problem can be generalized to show that if a function of a sufficient statistic is ancillary, then the sufficient statistic is not complete, because the expected value of that function does not depend on $\theta$. That provides the opportunity to construct an unbiased, nonzero estimator of zero.

---

6.11 a. These are all location families. Let $Z(1), \ldots, Z(n)$ be the order statistics from a random sample of size $n$ from the standard pdf $f(x|\theta)$. Then $(Z(1) + \theta, \ldots, Z(n) + \theta)$ has the same joint distribution as $(X(1), \ldots, X(n))$, and $(Y(1), \ldots, Y(n-1))$ has the same joint distribution as $(Z(n) + \theta - (Z(1) + \theta), \ldots, Z(n) + \theta - (Z(n-1) + \theta)) = (Z(n) - Z(1), \ldots, Z(n) - Z(n-1))$. The last vector depends only on $(Z_1, \ldots, Z_n)$ whose distribution does not depend on $\theta$. So, $(Y(1), \ldots, Y(n-1))$ is ancillary.

b. For a), Basu’s lemma shows that $(Y_1, \ldots, Y_{n-1})$ is independent of the complete sufficient statistic. For c), d), and e) the order statistics are sufficient, so $(Y_1, \ldots, Y_{n-1})$ is independent of the sufficient statistic. For b), $X(1)$ is sufficient. Define $Y_n = X(1)$. Then the joint pdf of $(Y_1, \ldots, Y_n)$ is

$$f(y_1, \ldots, y_n) = n!e^{-n(y_1 - \theta)} - (n-1)\theta \prod_{i=2}^{n-1} e^{\theta}, \quad 0 < y_{n-1} < y_n < \cdots < y_1 \quad 0 < y_n < \infty.$$

Thus, $Y_n = X(1)$ is independent of $(Y_1, \ldots, Y_{n-1})$.

---

6.12 a. Use Theorem 6.2.13 and write

$$\frac{f(x, n|\theta)}{f(y, n'|\theta)} = \frac{f(x|\theta, N = n)P(N = n)}{f(y|\theta, N = n')P(N = n')} = \frac{\binom{n}{x} \theta^x (1-\theta)^{n-x} n! p_n}{\binom{n'}{y} \theta^y (1-\theta)^{n'-y} n'! p_{n'}} = \theta^{x-y} (1-\theta)^{n-x+y} \left( \frac{n!}{y!} \right) \frac{p_n}{p_{n'}}.$$
The last ratio does not depend on \( \theta \). The other terms are constant as a function of \( \theta \) if and only if \( n = n' \) and \( x = y \). So \((X, N)\) is minimal sufficient for \( \theta \). Because \( P(N = n) = p_n \) does not depend on \( \theta \), \( N \) is ancillary for \( \theta \). The point is that although \( N \) is independent of \( \theta \), the minimal sufficient statistic contains \( N \) in this case. A minimal sufficient statistic may contain an ancillary statistic.

\[
\begin{align*}
\mathbb{E}(X / N) &= \mathbb{E}\left( \mathbb{E}(X / N) \mid N \right) = \mathbb{E}\left( \frac{1}{N} \mathbb{E}(X / N) \mid N \right) = \mathbb{E}\left( \frac{1}{N} N \theta \right) = \mathbb{E}(\theta) = \theta. \\
\text{Var}(X / N) &= \text{Var}\left( \mathbb{E}(X / N) \right) + \mathbb{E}\left( \text{Var}(X / N) \right) = \text{Var}(\theta) + \mathbb{E}\left( \frac{1}{N^2} \text{Var}(X / N) \right)
\end{align*}
\]

\[
= 0 + \mathbb{E}\left( \frac{N \theta (1 - \theta)}{N^2} \right) = \theta (1 - \theta) \mathbb{E}\left( \frac{1}{N} \right).
\]

We used the fact that \( X \mid N \sim \text{binomial}(N, \theta) \).

6.13 Let \( Y_1 = \log X_1 \) and \( Y_2 = \log X_2 \). Then \( Y_1 \) and \( Y_2 \) are iid and, by Theorem 2.1.5, the pdf of each is

\[
f(y|\alpha) = \alpha \exp\left\{ ay - e^{ay} \right\} = \frac{1}{a} \exp\left\{ \frac{y}{a} - e^{y/(1/a)} \right\}, \quad -\infty < y < \infty.
\]

We see that the family of distributions of \( Y_1 \) is a scale family with scale parameter \( 1/\alpha \). Thus, by Theorem 3.5.6, we can write \( Y_1 = \frac{1}{\alpha} Z_1 \), where \( Z_1 \) and \( Z_2 \) are a random sample from \( f(z) \mid 1 \). Then

\[
\frac{Y_1}{Y_2} = \frac{Z_1}{Z_2},
\]

because the distribution of \( Z_1 / Z_2 \) does not depend on \( \alpha \), \( (\log X_1) / (\log X_2) \) is an ancillary statistic.

6.14 Because \( X_1, \ldots, X_n \) is from a location family, by Theorem 3.5.6, we can write \( X_i = Z_i + \mu \), where \( Z_1, \ldots, Z_n \) is a random sample from the standard pdf, \( f(x) \), and \( \mu \) is the location parameter. Let \( M(X) \) denote the median calculated from \( X_1, \ldots, X_n \). Then \( M(X) = M(Z) + \mu \) and \( \bar{X} = \bar{Z} + \mu \). Thus, \( M(X) - \bar{X} = (M(Z) + \mu) - (\bar{Z} + \mu) = M(Z) - \bar{Z} \). Because \( M(X) - \bar{X} \) is a function of only \( Z_1, \ldots, Z_n \), the distribution of \( M(X) - \bar{X} \) does not depend on \( \mu \); that is, \( M(X) - \bar{X} \) is an ancillary statistic.

6.15 a. The parameter space consists only of the points \((\theta, \nu)\) on the graph of the function \( \nu = a \theta^2 \).

This quadratic graph is a line and does not contain a two-dimensional open set.

b. Use the same factorization as in Example 6.2.9 to show \((\bar{X}, S^2)\) is sufficient. \( E(S^2) = a \theta^2 \) and \( E(\bar{X}^2) = \text{Var} \bar{X} + E(\bar{X})^2 = a \theta^2 / n + \theta^2 = (a + n) \theta^2 / n \). Therefore,

\[
E\left( \frac{n}{a + n} \bar{X} - \frac{S^2}{n} \right) = \left( \frac{n}{a + n} \right) \left( \frac{a + n \theta^2}{n} - \frac{1}{a} a \theta^2 \right) = 0,
\]

for all \( \theta \).

Thus \( g(\bar{X}, S^2) = \frac{n}{a + n} \bar{X} - \frac{S^2}{a} \) has zero expectation so \((\bar{X}, S^2)\) not complete.

6.17 The population pmf is \( f(x|\theta) = \theta (1 - \theta) x^{-1} = \frac{\theta}{1 - \theta} \log(1 - \theta)x \), an exponential family with \( t(x) = x \). Thus, \( \sum_i X_i \) is a complete, sufficient statistic by Theorems 6.2.10 and 6.2.25. \( \sum_i X_i - n \sim \text{negative binomial}(n, \theta) \).

6.18 The distribution of \( Y = \sum_i X_i \) is Poisson\((n \lambda)\). Now

\[
E g(Y) = \sum_{y=0}^{\infty} g(y) \frac{(n \lambda)^y e^{-n \lambda}}{y!}.
\]

If the expectation exists, this is an analytic function which cannot be identically zero.
6.19 To check if the family of distributions of $X$ is complete, we check if $E_p g(X) = 0$ for all $p$, implies that $g(X) \equiv 0$. For Distribution 1,

$$E_p g(X) = \sum_{x=0}^{2} g(x) P(X = x) = pg(0) + 3pg(1) + (1 - 4p)g(2).$$

Note that if $g(0) = -3g(1)$ and $g(2) = 0$, then the expectation is zero for all $p$, but $g(x)$ need not be identically zero. Hence the family is not complete. For Distribution 2 calculate

$$E_p g(X) = g(0)p + (1)p^2 + g(2)(1 - p - p^2) = [g(1) - g(2)]p + g(0) - g(2)p + g(2).$$

This is a polynomial of degree 2 in $p$. To make it zero for all $p$ each coefficient must be zero. Thus, $g(0) = g(1) = g(2) = 0$, so the family of distributions is complete.

6.20 The pdfs in b), c), and e) are exponential families, so they have complete sufficient statistics from Theorem 6.2.25. For a), $Y = \max\{X_i\}$ is sufficient and

$$f(y) = \frac{2^n}{\theta^{2n}} y^{2n-1}, \quad 0 < y < \theta.$$ 

For a function $g(y)$,

$$E g(Y) = \int_0^\theta g(y) \frac{2^n}{\theta^{2n}} y^{2n-1} dy = 0 \quad \text{for all } \theta \Rightarrow g(0) = 0 \quad \text{for all } \theta$$

by taking derivatives. This can only be zero if $g(\theta) = 0$ for all $\theta$, so $Y = \max\{X_i\}$ is complete. For d), the order statistics are minimal sufficient. This is a location family. Thus, by Example 6.2.18 the range $R = X_{(n)} - X_{(1)}$ is ancillary, and its expectation does not depend on $\theta$. So this sufficient statistic is not complete.

6.21 a. $X$ is sufficient because it is the data. To check completeness, calculate

$$E g(X) = \frac{\theta}{2} g(-1) + (1 - \theta) g(0) + \frac{\theta}{2} g(1).$$

If $g(-1) = g(1)$ and $g(0) = 0$, then $E g(X) = 0$ for all $\theta$, but $g(x)$ need not be identically 0. So the family is not complete.

b. $|X|$ is sufficient by Theorem 6.2.6, because $f(x|\theta)$ depends on $x$ only through the value of $|x|$. The distribution of $|X|$ is Bernoulli, because $P(|X| = 0) = 1 - \theta$ and $P(|X| = 1) = \theta$. By Example 6.2.22, a binomial family (Bernoulli is a special case) is complete.

c. Yes, $f(x|\theta) = (1 - \theta)(\theta/(2(1 - \theta)))^{|x|} = (1 - \theta) e^{x \log(\theta/2(1 - \theta))}$, the form of an exponential family.

6.22 a. The sample density is $\prod_i \theta |x_i|^{\theta-1} = \theta^n (\prod_i |x_i|)^{\theta-1}$, so $\prod_i X_i$ is sufficient for $\theta$, not $\sum_i X_i$.

b. Because $\prod_i f(x_i|\theta) = \theta^{n \log(\theta^{-1})} (\prod_i x_i)^{\theta-1}$, $\log(\prod_i X_i)$ is complete and sufficient by Theorem 6.2.25. Because $\prod_i X_i$ is a one-to-one function of $\log(\prod_i X_i)$, $\prod_i X_i$ is also a complete sufficient statistic.

6.23 Use Theorem 6.2.13. The ratio

$$f(x|\theta) = \frac{e^{-n} I_{x(1)/2,x(1)}}{e^{-n} I_{y(1)/2,y(1)}}(\theta)$$

is constant (in fact, one) if and only if $x(1) = y(1)$ and $x(n) = y(n)$. So $(X(1), X(n))$ is a minimal sufficient statistic for $\theta$. From Exercise 6.10, we know that if a function of the sufficient statistics is ancillary, then the sufficient statistic is not complete. The uniform $(\theta, 2\theta)$ family is a scale family, with standard pdf $f(x) \sim \text{uniform}(1, 2)$. So if $Z_1, \ldots, Z_n$ is a random sample
6.29 Let \( f_j = \logistic(\alpha_j, \beta_j), \ j = 0, 1, \ldots, k. \) From Theorem 6.6.5, the statistic
\[
T(x) = \left( \prod_{i=1}^{n} f_1(x_i), \ldots, \prod_{i=1}^{n} f_k(x_i) \right) = \left( \prod_{i=1}^{n} f_1(x_i) \prod_{i=1}^{n} f_k(x_i) \right)
\]
is minimal sufficient for the family \( \{f_0, f_1, \ldots, f_k\}. \) As \( T \) is a \( 1 \times 1 \) function of the order statistics, the order statistics are also minimal sufficient for the family \( \{f_0, f_1, \ldots, f_k\}. \) If \( \mathcal{F} \) is a nonparametric family, \( f_j \in \mathcal{F}, \) so part (b) of Theorem 6.6.5 can now be directly applied to show that the order statistics are minimal sufficient for \( \mathcal{F}. \)

6.30 a. From Exercise 6.9b, we have that \( X_{(1)} \) is a minimal sufficient statistic. To check completeness compute \( f_{X_1}(y), \) where \( Y_1 = X_{(1)}. \) From Theorem 5.4.4 we have
\[
f_{X_1}(y) = f_X(y)(1 - F_X(y))^{n-1} = e^{-(y-\mu)} \left[ e^{-(y-\mu)} \right]^{n-1} = ne^{-(y-\mu)}, \quad y > \mu.
\]
Now, write \( E_{\mu} f(Y_1) = \int_{\mu}^{\infty} g(y)e^{-ny} dy. \) If this is zero for all \( \mu, \) then \( \int_{\mu}^{\infty} g(y)e^{-ny} dy = 0 \) for all \( \mu \) (because \( ne^{-\mu} > 0 \) for all \( \mu \) and does not depend on \( y \)). Moreover,
\[
0 = \frac{d}{d\mu} \int_{\mu}^{\infty} g(y)e^{-ny} dy = -g(\mu)e^{-\mu}
\]
for all \( \mu. \) This implies \( g(\mu) = 0 \) for all \( \mu, \) so \( X_{(1)} \) is complete.

b. Basu’s Theorem says that if \( X_{(1)} \) is a complete sufficient statistic for \( \mu, \) then \( X_{(1)} \) is independent of any ancillary statistic. Therefore, we need to show only that \( S^2 \) has distribution independent of \( \mu; \) that is, \( S^2 \) is ancillary. Recognize that \( f(x | \mu) \) is a location family. So we can write \( X_1 = Z_1 + \mu, \) where \( Z_1, \ldots, Z_n \) is a random sample from \( f(x | 0). \) Then
\[
S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2 = \frac{1}{n-1} \sum ((Z_i + \mu) - (\bar{Z} + \mu))^2 = \frac{1}{n-1} \sum (Z_i - \bar{Z})^2.
\]
Because \( S^2 \) is a function of only \( Z_1, \ldots, Z_n, \) the distribution of \( S^2 \) does not depend on \( \mu; \) that is, \( S^2 \) is ancillary. By Basu’s theorem, \( S^2 \) is independent of \( X_{(1)}). \)

6.31 a. (i) By Exercise 3.28 this is a one-dimensional exponential family with \( t(x) = x. \) By Theorem 6.2.25, \( \sum X_i \) is a complete sufficient statistic. \( \bar{X} \) is a one-to-one function of \( \sum X_i, \) so \( \bar{X} \) is also a complete sufficient statistic. From Theorem 5.3.1 we know that \( (n-1)S^2/\sigma^2 \sim \chi^2_{n-1} = \gamma((n-1)/2, 2). \) \( S^2 = [\sigma^2/(n-1)](n-1)S^2/\sigma^2, \) a simple scale transformation, has a gamma \((n-1)/2, 2\sigma^2/(n-1)\) distribution, which does not depend on \( \mu; \) that is, \( S^2 \) is ancillary. By Basu’s Theorem, \( \bar{X} \) and \( S^2 \) are independent.

(ii) The independence of \( \bar{X} \) and \( S^2 \) is determined by the joint distribution of \( (\bar{X}, S^2) \) for each value of \( (\mu, \sigma^2). \) By part (i), for each value of \( (\mu, \sigma^2), \) \( \bar{X} \) and \( S^2 \) are independent.

b. (i) \( \mu \) is a location parameter. By Exercise 6.14, \( M - \bar{X} \) is ancillary. As in part (a) \( \bar{X} \) is a complete sufficient statistic. By Basu’s Theorem, \( \bar{X} \) and \( M - \bar{X} \) are independent. Because they are independent, by Theorem 4.5.6 \( \text{Var} M = \text{Var}(M - \bar{X} + \bar{X}) = \text{Var}(M - \bar{X}) + \text{Var} \bar{X}. \)

(ii) If \( S^2 \) is a sample variance calculated from a normal sample of size \( N, (N-1)S^2/\sigma^2 \sim \chi^2_{N-1}. \) Hence, \( (N-1)\text{Var} S^2/(\sigma^2)^2 = 2(N-1) \) and \( \text{Var} S^2 = 2(\sigma^2)^2/(N-1). \) Both \( M \) and \( M - \bar{X} \) are asymptotically normal, so, \( M_1, \ldots, M_N \) and \( M_1 - \bar{X}_1, \ldots, M_N - \bar{X}_N \) are each approximately normal samples if \( n \) is reasonable large. Thus, using the above expression we get the two given expressions where in the straightforward case \( \sigma^2 \) refers to \( \text{Var} M, \) and in the swindle case \( \sigma^2 \) refers to \( \text{Var}(M - \bar{X}). \)

c. (i)
\[
E(X^k) = E\left( \frac{X}{\bar{X}} \right)^k = E\left( \frac{X}{\bar{X}} \right)^k \text{(by indep.)} = E\left( \frac{X}{\bar{X}} \right)^k E(Y^k).
\]
Divide both sides by \( E(Y^k) \) to obtain the desired equality.
(ii) If $\alpha$ is fixed, $T = \sum X_i$ is a complete sufficient statistic for $\beta$ by Theorem 6.2.25. Because $\beta$ is a scale parameter, if $Z_1, \ldots, Z_n$ are a random sample from a gamma$(\alpha, 1)$ distribution, then $X_i/\sum Z_i$ has the same distribution as $(\beta Z_i)/(\beta \sum Z_i) = Z_i/(\sum Z_i)$, and this distribution does not depend on $\beta$. Thus, $X_i/\sum Z_i$ is ancillary, and by Basu’s Theorem, it is independent of $T$. We have

$$E(X_i/\sum Z_i | T) = E\left(\frac{X_i}{T} \left| \frac{T}{\sum Z_i} \right. \right) = \frac{E(X_i)}{E(T)} = \frac{T E(X_i)}{E(T)}.$$

Note, this expression is correct for each fixed value of $(\alpha, \beta)$, regardless whether $\alpha$ is “known” or not.

6.32 In the Formal Likelihood Principle, take $E_1 = E_2 = E$. Then the conclusion is $E\{E(X_1) = E\{E(X_2) = c \}$. Thus evidence is equal whenever the likelihood functions are equal, and this follows from Formal Sufficiency and Conditionality.

6.33 a. For all sample points except $(2, x_2^2)$ (but including $(1, x_1^1)$), $T(j, x_j) = (j, x_j)$. Hence,

$$g(T(j, x_j)|\theta) h(j, x_j) = g((j, x_j)|\theta) 1 = f^*(j, x_j)|\theta)$$

For $(2, x_2^2)$ we also have

$$g(T(2, x_2^2)|\theta) h(2, x_2^2) = g((1, x_1^1)|\theta) C = f^*((1, x_1^1)|\theta) C = \frac{1}{2} f_1(x_1^1)|\theta)$$

$$= \frac{1}{2} \frac{1}{2} L(\theta|x_1^1) = \frac{1}{2} f_2(x_2^2|x_1^1) = f^*((2, x_2^2)|\theta).$$

By the Factorization Theorem, $T(J, X_J)$ is sufficient.

b. Equations 6.3.4 and 6.3.5 follow immediately from the two Principles. Combining them we have $E\{E_1(x_1^1) = E\{E_2(x_2^1) = E\{x_1^1 = E\{x_2^1\}$, the conclusion of the Formal Likelihood Principle.

c. To prove the Conditionality Principle. Let one experiment be the $E^*$ experiment and the other $E_j$. Then

$$L(\theta|x_1, x_j) = f^*(j, x_j|\theta) = \frac{1}{2} f_j(x_j|\theta) = \frac{1}{2} L(\theta|x_j).$$

Letting $(j, x_j)$ and $x_j$ play the roles of $x_1^1$ and $x_2^2$ in the Formal Likelihood Principle we can conclude $E\{E^*(j, x_j) = E\{E_1(x_1^1) = E\{x_1^1 = E\{x_2^1\}$, the Conditionality Principle. Now consider the Formal Sufficiency Principle. If $T(X)$ is sufficient and $T(x) = T(y)$, then $L(\theta|x) = C L(\theta|y)$, where $C = h(x)/h(y)$ and $h$ is the function from the Factorization Theorem. Hence, by the Formal Likelihood Principle, $E\{E_1(x_1^1) = E\{E_2(x_2^2) = E\{x_1^1 = E\{x_2^2\}$, the Formal Sufficiency Principle.

6.35 Let $i = \text{success}$ and $0 = \text{failure}$. The four sample points are $\{0, 0, 1, 1, 11, 11\}$. From the likelihood principle, inference about $p$ is only through $L(p|x)$. The values of the likelihood are $1, p, p^2$, and $p^3$, and the sample size does not directly influence the inference.

6.37 a. For one Observation $(X, Y)$ we have

$$I(\theta) = -E \left( \frac{\partial^2}{\partial \theta^2} \log f(X, Y|\theta) \right) = -E \left( \frac{-2Y}{\theta^2} \right) = \frac{2EY}{\theta^2}.$$

But, $Y \sim \text{exponential}(\theta)$, and $EY = \theta$. Hence, $I(\theta) = 2/\theta^2$ for a sample of size one, and $I(\theta) = 2n/\theta^2$ for a sample of size $n$.

b. (i) The cdf of $T$ is

$$P(T \leq t) = P\left( \sum \frac{Y_i}{\sum X_i} \leq t^2 \right) = P\left( \frac{2 \sum Y_i / \theta}{2 \sum X_i / \theta} \leq t^2 / \theta^2 \right) = P(F_{2n, 2n} \leq t^2 / \theta^2)$$