6.2

Calculate

$$F_Y(y) = \int_{-1}^{y} \tfrac{3}{2}t^2\,dt = \tfrac{1}{2}(y^3+1)$$

for $-1 \le y \le 1$

a. $F_{U_1}(u) = P(3Y \le u) = P\left(Y \le \frac{u}{3}\right) = F_Y\left(\frac{u}{3}\right) = \frac{1}{2}\left(\frac{u^3}{27}+1\right)$ for $-3 \le u \le 3$

Differentiating with respect to u, we have

$$f_{U_1}(u) = \begin{cases} \frac{u^2}{18}, & -3 \le u \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

b. $F_{U_2}(u) = P(3 - Y \le u) = P(Y \ge 3-u) = 1 - F_Y(3-u) = \frac{1}{2}\left[1-(3-u)^3\right]$ for $-1 \le (3-u) \le 1$ or $2 \le u \le 4$. Differentiating with respect to u, we have

$$f_{U_2}(u) = \begin{cases} \left(\frac{3}{2}\right)(3-u)^2, & 2 \le u \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

c. $F_{U_3}(u) = P(Y^2 \le u) = P\left(-\sqrt{u} \le Y \le \sqrt{u}\right) = F_Y\left(\sqrt{u}\right) - F_Y\left(-\sqrt{u}\right)$
$= \frac{1}{2}u^{3/2} + \frac{1}{2}u^{3/2} = u^{3/2}$ for $0 \le y^2 \le 1$ or $0 \le u \le 1$.

Hence

$$f_{U_3}(u) = \tfrac{3}{2}u^{1/2} = \tfrac{3}{2}\sqrt{u}$$

for $0 \le u \le 1$

6.10

If Y_1 and Y_2 are independent, then $f(y_1, y_2) = 18\left(y_1 - y_1^2\right)y_2^2$ for $0 \le y_1 \le 1$, $0 \le y_2 \le 1$. Using the distribution function approach, we see that $F_U(u)$ is the unshaded area in Figure 6.5. Integrating the shaded area and subtracting from 1, we have

$$\begin{aligned} F_U(u) &= P(Y_1Y_2 \le u) \\ &= 1 - 18\int_u^1 \int_{u/y_2}^1 \left(y_1 - y_1^2\right)y_2^2\,dy_1\,dy_2 \\ &= 1 - 18\int_u^1 \left(\frac{y_2^2}{6} - \frac{u^2}{2} + \frac{u^3}{3y_2}\right)dy_2 \\ &= 9u^2 - 8u^3 + 6u^3\ln u \qquad \text{for } 0 \le u \le 1 \end{aligned}$$

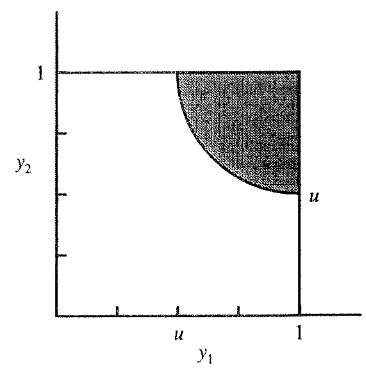

Figure 6.5

Differentiating with respect to u, we have

$$\begin{aligned} f_U(u) &= 18u - 24u^2 + \frac{6u^3}{u} + 18u^2\ln u = 18u(1 - u + u\ln u) && \text{for } 0 \le u \le 1 \\ &= 0 && \text{elsewhere} \end{aligned}$$

6.12

This is similar to Exercise 6.11. From Exercise 4.7, $f(y) = \frac{b}{y^2}$, $y \ge b$ so that

$$F_Y(y) = b\int_b^y t^{-2}\,dt = 1 - \frac{b}{y},\ y \ge \text{b}.$$

Let $Y = G(U)$ so that $U = G^{-1}(Y)$

$$F_Y(y) = P(G(U) \le y) = P(U \le u) = u,\ 0 < u < 1.$$

We need $u = 1 - \frac{b}{y}$ or $y = \frac{b}{1-u}$. Thus, $G(U) = \frac{b}{1-U}$.