10.82 We begin by looking at the most powerful test for H_0: $\sigma^2 = \sigma_0^2$ vs. H_a: $\sigma^2 = \sigma_1^2$ for $\sigma_1^2 > \sigma_0^2$. The null hypothesis specifies that $\sigma^2 = \sigma_0^2$, so that

$$L(\sigma_0^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-(y_i-\mu)^2/2\sigma_0^2} = \frac{1}{\left(\sqrt{2\pi}\right)^n \sigma_0^n} \exp\left[\frac{-\sum (y_i-\mu)^2}{2\sigma_0^2}\right].$$

Similarly,

$$L(\sigma_a^2) = \frac{1}{\left(\sqrt{2\pi}\right)^n \sigma_1^n} \exp\left[\frac{-\sum (y_i-\mu)^2}{2\sigma_1^2}\right].$$

The most powerful test is

$$\frac{L(\sigma^2)}{L(\sigma_a^2)} = \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left[-\frac{\sigma_1^2-\sigma_0^2}{2\sigma_1^2\sigma_0^2} \sum (y_i-\mu)^2\right] \le k.$$

Taking natural logarithms, we have

$$n \ln\left(\frac{\sigma_1}{\sigma_0}\right) - \left(\frac{\sigma_1^2-\sigma_0^2}{2\sigma_1^2\sigma_0^2}\right) \sum (y_i-\mu)^2 \le \ln k$$

or

$$\sum (y_i-\mu)^2 \ge \left[n \ln\left(\frac{\sigma_1}{\sigma_0}\right) - \ln k\right] \left(\frac{2\sigma_1^2\sigma_0^2}{\sigma_1^2-\sigma_0^2}\right) = c.$$

To find the rejection region for a fixed α, write the region as

$$\frac{\sum (y_i-\mu)^2}{\sigma_0^2} \ge \frac{c}{\sigma_0^2} = c'$$

and note that $\frac{\sum (y_i-\mu)^2}{\sigma_0^2}$ has a χ^2 distribution with n degrees of freedom. Since the same rejection region would be used for any $\sigma_1 > \sigma_0$, the test is uniformly most powerful.

10.99 a. Under the null hypothesis, with $\Omega_0 = \{\theta_0\}$, the likelihood is maximized at θ_0. Then, under the alternative hypothesis, with $\Omega = \{\theta_0, \theta_a\}$, the likelihood is maximized at either θ_0 or θ_a. Thus, $L(\widehat{\Omega}_0) = L(\theta_0)$ and $L(\widehat{\Omega}) = \max\{L(\theta_0), L(\theta_a)\}$, so that

$$\lambda = \frac{L(\widehat{\Omega}_0)}{L(\widehat{\Omega})} = \frac{L(\theta_0)}{\max\{L(\theta_0), L(\theta_a)\}} = \frac{1}{\max\left\{1, \frac{L(\theta_a)}{L(\theta_0)}\right\}}$$

b. First, recognize that

$$\lambda = \frac{1}{\max\left\{1, \frac{L(\theta_a)}{L(\theta_0)}\right\}} = \min\left\{1, \frac{L(\theta_0)}{L(\theta_a}\right\}.$$

Now, as mentioned in Example 10.24, we restrict the attention to $k < 1$. Then

$$\lambda < k$$

if and only if

$$\min\left\{1, \frac{L(\theta_0)}{L(\theta_a}\right\} < k < 1$$

if and only if

$$\frac{L(\theta_0)}{L(\theta_a)} < k.$$

c. These results imply that in the case of both simple null and alternative hypotheses, the likelihood ratio test is equivalent to the most powerful test as given by the Neymann–Pearson Lemma.

b.

11.8

a. We calculate:

$\sum_{i=1}^{5} x_i = 102$ $\qquad \sum_{i=1}^{5} x_i^2 = 3940$ $\qquad \sum_{i=1}^{5} y_i = 64.7$

$\sum_{i=1}^{5} x_i y_i = 894.4$ $\qquad \sum_{i=1}^{5} y_i^2 = 949.99$ $\qquad n = 5$

$S_{xy} = -425.48$ $\qquad S_{xx} = 1859.2$

Implying

$$\widehat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = -.229$$

and

$$\widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1 \overline{x} = \frac{64.7}{5} - (-.229)\left(\frac{102}{5}\right) = 17.611.$$

b. The data and least squares line are plotted in Figure 11.4.