6.78 Let Y_1 be the point on the one-mile stretch chosen for sentry 1 and let Y_2 be the point chosen for sentry 2. It is given that $f_1(y_1) = 1$ for $0 \le y_1 \le 1$ and $f_2(y_2) = 1$ for $0 \le y_2 \le 1$. Since Y_1 and Y_2 are independent, their joint distribution is

$$f(y_1, y_2) = f_1(y_1)f_2(y_2) = 1 \qquad 0 \le y_1 \le 1, 0 \le y_2 \le 1$$

The distance between the two sentries, which is the variable of interest, can be represented by $|Y_1 - Y_2|$. Let us consider first the joint distribution of $u = Y_1 - Y_2$ and Y_2. Letting $Y_2 = y_2$, we have $Y_1 = U + y_2$ and $\left|\frac{dy_1}{du}\right| = 1$. Hence $g(y_2, u) = 1$. The region on which the density is positive is shown in Figure 6.10.

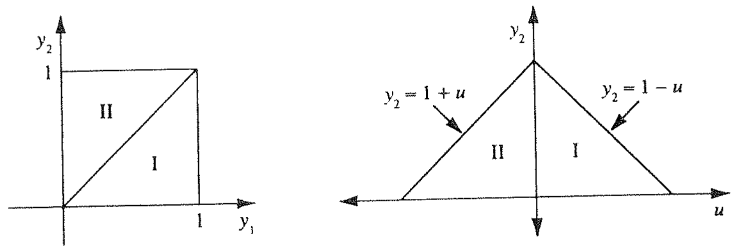

Figure 6.10

$$g(u) = \int_0^{1+u} dy_2 = 1 + u \qquad \text{for} \qquad -1 \le u \le 0$$

$$g(u) = \int_0^{1-u} dy_2 = 1 - u \qquad \text{for} \qquad 0 \le u \le 1.$$

Now the probability of interest is

$$P\left(|Y_1 - Y_2| < \tfrac{1}{2}\right) = P\left(|U| < \tfrac{1}{2}\right) = P\left(-\tfrac{1}{2} \le U \le \tfrac{1}{2}\right)$$

$$= \int_{-1/2}^{0} (1+u)\, du + \int_0^{1/2} (1-u)\, du$$

$$= \frac{3}{4}$$

7.2 **a.** $P(|\overline{Y}-\mu| \le .3) = P\left[-.3 \le (\overline{Y}-\mu) \le .3\right] = P\left(\frac{-.3}{\left(\frac{\sigma}{\sqrt{n}}\right)} \le \frac{Y-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \le \frac{.3}{\left(\frac{\sigma}{\sqrt{n}}\right)}\right)$
$= P\left(-.3\sqrt{n}/2 \le z \le .3\sqrt{n}/2\right) = P\left(-.15\sqrt{n} \le z \le .15\sqrt{n}\right)$
$= 1 - 2P(z > .15\sqrt{n}) = 1 - 2P(z > .45) = 1 - 2(.3264) = .3472$. This probability is less than that found in example 7.1 (as σ has increased).

b. The probability in question is $1 - 2P(z > .15\sqrt{n})$ which is $1 - 2P(z > .75) = .5468$ for $n = 25$, $.6318$ for $n = 36$, $.7062$ for $n = 49$, and $.7698$ for $n = 64$.

c. The probabilities increase as n increases.

d. As we increase σ the variability of $\overline{Y}$ increases. Therefore the probabilities for exercise 7.2 part **b** should be smaller than those for exercise 7.1 part **b.** (which is exactly what we see).

7.10 **a.** Since a χ^2 random variable is defined as a gamma random variable with $\alpha = \frac{\nu}{2}$ and $\beta = 2$, the expected value and variance are

$$E(U) = \alpha\beta = \nu \qquad \text{and} \qquad V(U) = \alpha\beta^2 = 2\nu$$

b. Using Theorem 7.3, we see that the quantity $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2 distribution with $\nu = n - 1$. Hence, from part **a**,

$$E\left(\frac{(n-1)S^2}{\sigma^2}\right) = n - 1 \quad \text{or} \quad \left(\frac{n-1}{\sigma^2}\right) E\left(S^2\right) = n - 1 \quad \text{or} \quad E\left(S^2\right) = \sigma^2$$

Similarly,

$$V\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1) \quad \text{or} \quad \left[\frac{(n-1)^2}{\sigma^4}\right] V\left(S^2\right) = 2(n-1) \quad \text{or} \quad V\left(S^2\right) = \frac{2\sigma^4}{n-1}$$

7.12 Similar to Exercise 7.11. Since, from Definition 7.2,

$$P\left(g_1 \le (\overline{Y}-\mu) \le g_2\right) = P\left(\frac{g_1}{\left(\frac{s}{\sqrt{n}}\right)} \le t \le \frac{g_2}{\left(\frac{s}{\sqrt{n}}\right)}\right)$$

the necessary probability will be obtained if we take

$$\frac{g_1}{\left(\frac{s}{\sqrt{n}}\right)} = -t_{.05} \qquad \text{and} \qquad \frac{g_2}{\left(\frac{s}{\sqrt{n}}\right)} = t_{.05}$$

Indexing $n - 1 = 8$ degrees of freedom, we have $t_{.05} = 1.86$, $g_1 = \left(\frac{-1.86}{3}\right) S$, and $g_2 = \left(\frac{1.86}{3}\right) S$.