HOMEWORK #4

7.18 a. Assume that $\sigma_1^2 = 2\sigma_2^2$, with $n_1 = 10$ and $n_2 = 8$. Using Definition 7.3 and Theorem 7.3, we know that

$$\frac{\left(\frac{S_1^2}{\sigma_1^2}\right)}{\left(\frac{S_2^2}{\sigma_2^2}\right)} = \frac{\left(\frac{S_1^2}{2\sigma_2^2}\right)}{\left(\frac{S_2^2}{\sigma_2^2}\right)} = \frac{S_1^2}{2S_2^2}$$

has an F distribution with $n_1 - 1 = 9$ and $n_2 - 1 = 7$ degrees of freedom. Therefore $.95 = P\left(\frac{S_1^2}{S_2^2} \le b\right) = P\left(\frac{S_1^2}{2S_2^2} \le \frac{b}{2}\right)$. By setting $\frac{b}{2} = F_{.05} = 3.68$ we get $b = 7.36$.

b. Now we want $.95 = P\left(a \le \frac{S_1^2}{S_2^2}\right) = P\left(\frac{a}{2} \le \frac{S_1^2}{2S_2^2}\right)$. Therefore we set $\frac{a}{2} = F_{.95}$.

We note that $F_{.95}$ satisfies $P\left(F_{.95} \le \frac{S_1^2}{2S_2^2}\right) = .95$, implying $P\left(\frac{S_1^2}{2S_2^2} < F_{.95}\right) = .05$.

Now we use the fact that $\frac{2S_2^2}{S_1^2}$ has an F distribution with $n_2 - 1 = 7$ and $n_1 - 1 = 9$ degrees of freedom, and

$$P\left(\frac{S_1^2}{2S_2^2} < F_{.95}\right) = P\left(\frac{2S_2^2}{S_1^2} > \frac{1}{F_{.95}}\right) = .05$$

Then we have

$$\frac{1}{F_{.95}} = 3.29 \qquad \text{or} \qquad F_{.95} = \frac{1}{3.29} = .304.$$

Therefore $a = .304(2) = .608$.

c.
$$\begin{aligned}
P\left(a \le \frac{S_1^2}{S_2^2} \le b\right) &= 1 - P\left(a > \frac{S_1^2}{S_2^2}\right) - P\left(\frac{S_1^2}{S_2^2} > b\right) \\
&= 1 - \left(1 - P\left(a \le \frac{S_1^2}{S_2^2}\right)\right) - \left(1 - P\left(\frac{S_1^2}{S_2^2} \le b\right)\right) \\
&= 1 - (1 - .95) - (1 - .95) = 1 - .05 - .05 = .90.
\end{aligned}$$

7.38 It is given that $X_1, X_2, \ldots, X_n$ are independent and identically distributed with $E(X_i) = \mu_1$ and $V(X_i) = \sigma_1^2$. Similarly, $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed with $E(Y_i) = \mu_2$ and $V(Y_i) = \sigma_2^2$. Consider $d_i = X_i - Y_i$, for $i = 1, 2, \ldots, n$. The d_i's are independent and identically distributed with $E(d_i) = E(X_i) - E(Y_i) = \mu_1 - \mu_2$ and $V(d_i) = V(X_i) + V(Y_i) = \sigma_1^2 + \sigma_2^2 < \infty$. Hence, applying Theorem 7.4 to the set $d_1, d_2, \ldots, d_n$, we have

$$Y_n = \frac{[\bar{d} - (\mu_1 - \mu_2)]\sqrt{n}}{\sqrt{\sigma_1^2 + \sigma_2^2}} = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}}}$$

7.66 The desired probability is

$$P\left(|\bar{Y}_1 - \bar{Y}_2| > .6\right) = P\left(|Z| > \frac{.6 - 0}{\sqrt{\frac{(6.4)^2}{64} + \frac{(7.2)^2}{64}}}\right) = P(|Z| > .50) = 2(.30$$

8.8 a. For the uniform distribution given here, $E(Y_i) = \theta + \frac{1}{2}$. Hence $E(\bar{Y}) = \theta + \frac{1}{2}$ and the bias is $B = E(\bar{Y}) - \theta = \frac{1}{2}$.

b. An unbiased estimator of θ can be constructed by using $\hat{\theta} = \bar{Y} - \frac{1}{2}$, which has

$$E(\hat{\theta}) = \theta.$$

c. If $\bar{Y}$ is used as an estimator, then

$$V(\bar{Y}) = \frac{V(Y)}{n} = \frac{1}{12n} \qquad \text{and} \qquad \text{MSE} = V(\bar{Y}) + B^2 = \frac{1}{12n} + \frac{1}{4}.$$