HOMEWORK #7

8.114 The width of the small-sample confidence interval is $2t_{\alpha/2}\left(\frac{S}{\sqrt{n}}\right)$ and has expected value $2t_{\alpha/2}\left[\frac{E(S)}{\sqrt{n}}\right]$. From Exercise 8.12,

$$E(S) = \frac{\Gamma\left(\frac{n}{2}\right)\sqrt{2}\,\sigma}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{n-1}}$$

so that

$$E\left(2t_{\alpha/2}\left(\frac{S}{\sqrt{n}}\right)\right) = 2^{3/2}\,t_{\alpha/2}\,\frac{\sigma\Gamma\left(\frac{n}{2}\right)}{\sqrt{n(n-1)}\,\Gamma\left(\frac{n-1}{2}\right)}$$

9.2

a. Since $E(Y_i) = \mu$, $i = 1, 2, \ldots, n$, then

$E(\hat{\mu}_1) = \frac{1}{2}(2\mu) = \mu \qquad E(\hat{\mu}_2) = \frac{1}{4}\mu + \frac{(n-2)\mu}{2(n-2)} + \frac{1}{4}\mu = \mu \qquad E(\hat{\mu}_3) = \frac{n\mu}{n} = \mu$

b. Further,

$V(\hat{\mu}_1) = \frac{1}{4}(2\sigma^2) = \frac{\sigma^2}{2} \qquad V(\hat{\mu}_2) = \frac{2}{16}\sigma^2 + \frac{(n-2)\sigma^2}{4(n-2)^2} = \frac{\sigma^2}{8} + \frac{\sigma^2}{4(n-2)} \qquad V(\hat{\mu}_3) = \frac{\sigma^2}{n}$

Hence the efficiency of $\hat{\mu}_3$ relative to $\hat{\mu}_1$ is

$$\frac{V(\hat{\mu}_1)}{V(\hat{\mu}_3)} = \frac{\left(\frac{\sigma^2}{2}\right)}{\left(\frac{\sigma^2}{n}\right)} = \frac{n}{2}$$

and the efficiency of $\hat{\mu}_3$ relative to $\hat{\mu}_2$ is

$$\frac{V(\hat{\mu}_2)}{V(\hat{\mu}_3)} = \frac{\frac{\sigma^2}{8} + \frac{\sigma^2}{4(n-2)}}{\frac{\sigma^2}{n}} = \frac{n}{8} + \frac{n}{4(n-2)} = \frac{n^2}{8(n-2)}$$

9.13 Given $f(y)$, calculate

$$E(Y) = \frac{\theta}{\theta+1}$$

and

$$E(Y^2) = \frac{\theta}{\theta+2}.$$

Thus

$$V(Y) = \frac{\theta}{\theta+2} - \frac{\theta^2}{(\theta+1)^2} = \frac{\theta}{(\theta+2)(\theta+1)^2}$$

Hence, $E(\overline{Y}) = \frac{\theta}{\theta+1}$. Since $\sigma^2 = V(Y)$ is finite, the law of large numbers (Example 9.2) holds, and $\overline{Y}$ is consistent for $\frac{\theta}{\theta+1}$.

9.31 Refer to Definition 9.3. Each Y_i has a Poisson distribution with mean λ; hence ΣY_i has a Poisson distribution with mean $n\lambda$. The conditional distribution of $Y_1, Y_2, \ldots, Y_n$ given ΣY_i is

$$P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n | \Sigma Y_i = x) = \frac{p(Y_1=y_1, Y_2=y_2, \ldots, Y_n=y_n)}{P(\Sigma Y_i = x)}$$

$$= \frac{\prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}}{P(\Sigma Y_i = x)} = \frac{\left(\frac{e^{-n\lambda}\lambda^{\Sigma y_i}}{\prod_{i=1}^{n} y_i!}\right)}{\left(\frac{e^{-n\lambda}\lambda^{\Sigma y_i}}{(\Sigma y_i)!}\right)} = \begin{cases} \frac{(\Sigma y_i)!}{\prod_{i=1}^{n} y_i!}, & \text{if } \sum_{i=1}^{n} y_i = x \\ 0, & \text{otherwise} \end{cases}$$

which is independent of λ. Hence ΣY_i is sufficient for λ.