HOMEWORK #8

9.75 a. The likelihood function is

$$L = \prod_{i=1}^{n} \frac{1}{(2\theta+1)} I(0 < y_i < 2\theta + 1) = \frac{1}{(2\theta+1)^n} I(0 < y_{(n)} < 2\theta + 1)$$
$$= \frac{1}{(2\theta+1)^n} I\left(-\tfrac{1}{2} < \tfrac{1}{2}(y_{(n)} - 1) < \theta\right)$$

Notice that $\frac{1}{(2\theta+1)^n}$ is a decreasing function of θ on the interval $(-\frac{1}{2}, \infty)$. As $-\frac{1}{2} < \frac{1}{2}(y_{(n)} - 1) < \theta$ we know that the maximum must occur at $\hat{\theta} = \left(\frac{1}{2}\right)(Y_{(n)} - 1)$.

b. The variance of the distribution is $\frac{(2\theta+1)^2}{12}$. Since $\theta > 1$, this is a one-to-one function. Thus, the MLE for $\frac{(2\theta+1)^2}{12}$ is

$$\frac{(2\hat{\theta}+1)^2}{12} = \frac{\left[2\left(\frac{1}{2}\right)(Y_{(n)}-1)+1\right]^2}{12} = \frac{Y_{(n)}^2}{12}.$$

9.94 Using the method given in Exercise 9.49, we construct a random variable T, where $T = 1$ if $Y_1 = 0$ and $T = 0$ otherwise. Then

$$E(T) = P(Y_1 = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda}$$

and T is unbiased for $e^{-\lambda}$. Then since ΣY_i is sufficient for λ and $e^{-\lambda}$, we calculate $E(T|\Sigma Y_i)$, which will be the MVUE for $e^{-\lambda}$. Recall that ΣY_i has a Poisson distribution with mean $n\lambda$.

$$E\left(T|\sum_{i=1}^{n} Y_i\right) = P\left(T = 1|\sum_{i=1}^{n} Y_i = x\right) = P\left(Y_1 = 0|\sum_{i=1}^{n} Y_i = x\right)$$
$$= \frac{P\left(Y_1=0, \sum_{i=1}^{n} Y_i=x\right)}{P\left(\sum_{i=1}^{n} Y_i=x\right)} = \frac{P\left(Y_1=0, \sum_{i=2}^{n} Y_i=x\right)}{P\left(\sum_{i=1}^{n} Y_i=x\right)}$$
$$= \frac{e^{-\lambda}\left(e^{-(n-1)\lambda}\frac{[(n-1)\lambda]^x}{x!}\right)}{\frac{e^{-n\lambda}(n\lambda)^x}{x!}} = \frac{e^{-\lambda}e^{\lambda}(n-1)^x}{n^x} = \left(1 - \frac{1}{n}\right)^x = \left(1 - \frac{1}{n}\right)^{\Sigma Y_i}$$

Note that we made use of the fact that, if $Y_i = 0$, the sum $\Sigma Y_i = x$ must be obtained on the other $(n-1)$ trials, and that the sum of $(n-1)$ Poisson random variables has a Poisson distribution with mean $(n-1)\lambda$.