Homework 12 – do any three of the following questions

(this is also a practice final)

1. Suppose a 1-sample t-test of H_0: $\mu=0$ versus H_A: $\mu\neq 0$ yields a p-value of 0.004. What exactly does this mean? Suppose (0.5,2.3) is a 95% confidence interval for μ. What does this mean?

2. Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample from an **exponential** distribution with mean θ. Recall that the exponential(θ) probability density function (pdf) is given by

$$f_Y(y) = \begin{cases} \frac{1}{\theta}e^{-y/\theta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

and that the cumulative distribution function (cdf) is given by

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-y/\theta}, & y > 0. \end{cases}$$

Consider the **two estimators**

$$\widehat{\theta}_1 = nY_{(1)} \qquad \widehat{\theta}_2 = \overline{Y},$$

where $Y_{(1)}$ is the minimum of $Y_1, Y_2, ..., Y_n$, and $\overline{Y}$ is the sample mean. Recall that the pdf of the minimum order statistic $Y_{(1)}$, where positive, is given by

$$f_{Y_{(1)}}(y) = nf_Y(y)[1 - F_Y(y)]^{n-1}.$$

(a) Show that $Y_{(1)}$ follows an exponential distribution with mean θ/n.
(b) Show that both $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are unbiased estimators of θ.
(c) Compute the variance of both estimators. Which estimator would you prefer?

3. Suppose that $Z_1, Z_2, ..., Z_6$ is an iid sample from a $\mathcal{N}(0,1)$ distribution. Suppose that $Z_7 \sim \mathcal{N}(0,1)$ as well and that Z_7 is independent of $Z_1, Z_2, ..., Z_6$. **For each of the following parts, provide sufficient justification (in words) as to why your answer is correct.**

Find the distribution of

(a) $\overline{Z} = \frac{1}{6}\sum_{i=1}^{6} Z_i$

(b) $T = \sum_{i=1}^{6}(Z_i - \overline{Z})^2$

(c) $U = \sqrt{3}Z_7/\sqrt{Z_1^2 + Z_2^2 + Z_3^2}$

4. Suppose that $Y_1, Y_2, \ldots, Y_n$ is an iid sample from a **Poisson** distribution with mean θ.

(a) Give **two** unbiased estimators of θ.
(b) Find an unbiased estimator of $\tau(\theta) = \theta^2$.

5. Suppose that $X_1, \ldots, X_n$ form n Bernoulli trials with parameter $\theta = (1/3)(1+\beta)$, where the value of β is unknown ($0 <= \beta <= 1$). Determine the maximum likelihood estimator of β.

6. Suppose you have a random sample from a normally distributed population with mean μ and variance 1. What are the steps you would follow to construct a hypothesis test that the population 75th percentile is 2 versus the alternative hypothesis that the population 75th percentile is equal to 3? Would your test be a most powerful test?

7. The lifetime of a certain brand of light bulb, Y, is assumed to follow a gamma distribution with $\alpha = 3$ and $\beta = 6$. A random sample of n = 100 light bulbs is available. Approximate the probability that the sample mean Ybar will be larger than 21.