# PERMANENTAL PARTITION MODELS AND MARKOVIAN GIBBS STRUCTURES

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Abstract. We study both time-invariant and time-varying Gibbs distributions for configurations of particles into disjoint clusters. Specifically, we introduce and give some fundamental properties for a class of partition models, called *permanental partition models*, whose distributions are expressed in terms of the $\alpha$-permanent of a similarity matrix parameter. We show that, in the time-invariant case, the permanental partition model is a refinement of the celebrated Pitman-Ewens distribution; whereas, in the time-varying case, the permanental model refines the Ewens cut-and-paste Markov chains [*J. Appl. Probab.* (2011), **3**:778–791]. By a special property of the $\alpha$-permanent, the partition function can be computed exactly, allowing us to make several precise statements about this general model, including a characterization of exchangeable and consistent permanental models.

## 1. Introduction

Consider a collection of $n$ distinct particles (labeled $i = 1, \ldots, n$) located at distinct positions $x_1, \ldots, x_n$ in some space $\mathcal{X}$. Each particle $i = 1, \ldots, n$ occupies an *internal state* $y_i$, which may change over time. Jointly, the state of the system at time $t \geq 0$ depends on the positions of the particles $x_1, \ldots, x_n$ and the interactions among these particles. We assume that, with the exception of their positions in $\mathcal{X}$, particles have identical physical properties and, thus, the pairwise interaction between particles at positions $x$ and $x'$ is determined only by the pair $(x, x')$. In this setting, we model pairwise interactions by a symmetric function $\mathbf{K} : \mathcal{X} \times \mathcal{X} \to [0, 1]$ with unit diagonal, $\mathbf{K}(x, x) = 1$ for all $x \in \mathcal{X}$, where $\mathbf{K}(x, x')$ quantifies the interaction between particles at locations $x$ and $x'$. Qualitatively, we regard $\mathbf{K}$ as the *environment*, or *medium*, equipped on $\mathcal{X}$. For a fixed configuration $x_1, \ldots, x_n$, we simply represent the interactions among $x_1, \ldots, x_n$ by a matrix $\mathbf{X} := (X_{ij},\ 1 \leq i, j \leq n)$ for which $X_{ij} := \mathbf{K}(x_i, x_j)$. For example, the Ising model describes a system with internal states $\mathcal{Y} = \{\pm 1\}$ and interactions determined by the adjacency matrix of a weighted graph. More generally, the above scheme describes an interacting particle system.

In this paper, we consider interacting particle systems where the configuration is a partition of the particles $1, \ldots, n$ into disjoint clusters. In particular, we study a family of Gibbs measures on partitions of a finite set. Previously, Gibbs measures for fragmentation processes have appeared [4] in the context of modeling degradation of mass over time. More generally, models of agglomeration and fragmentation have been studied under the heading of coagulation and fragmentation processes, see e.g. [3, 5, 6, 33]. The exchangeable, or mean-field, case has garnered special attention; and, in our setting, the mean-field case

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corresponds to an environment that is *uniformly homogeneous*, i.e. $\mathbf{K}(x, x') \equiv 1$ for all $x, x' \in \mathcal{X}$; see Aldous [1] for a review of mean-field models for stochastic coalescence.

Given an *energy function* $H : \mathcal{Y} \to \mathbb{R}$ and a *temperature* $\beta \in \mathbb{R}$, a Gibbs distribution on $\mathcal{Y}$ has the form

$$P_{\beta,H}\{Y = y\} = \frac{1}{Z_{\beta,H}} \exp\left\{\frac{1}{\beta} H(y)\right\}, \quad y \in \mathcal{Y}. \tag{1}$$

Boltzmann-Gibbs distributions arise in various settings in both statistical mechanics and elsewhere. In the special case where $\mathcal{Y}$ is the collection of partitions of $[n] := \{1, \ldots, n\}$, the energy function $H$ can be assumed to have the *product partition form*; that is, for any partition $\pi$, $H(\pi)$ can be expressed as a sum over the blocks of $\pi$. Product partition models were initially introduced by Hartigan [18] in the context of astronomical modeling and have since appeared in several applications involving partition models, including Bayesian nonparametrics. Because of its general form, many common distributions are of product partition type, including the Ewens distribution [14] from population genetics; however, the more general Pitman-Ewens distributions [33] are not strictly of product partition type. In this paper, we modify (1) so as to include both the Pitman-Ewens distributions and the more general permanental partition model, our principle model of interest.

Extending (1) to a time-varying model, we define a *Markovian Gibbs structure* on $\mathcal{Y}$ as a transition probability of the form

$$p_{\beta,H}(y, y') = \frac{1}{Z_{\beta,H}(y)} \exp\left\{\frac{1}{\beta} H(y, y')\right\}, \quad y, y' \in \mathcal{Y}, \tag{2}$$

where $H : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ is called the *joint energy function* and $p_{\beta,H}(y, y')$ gives the probability of a transition from $y$ to $y'$. In specializing to Markovian Gibbs structures on partitions, we restrict $H$ to functions of $(\pi, \pi')$ that can be expressed as a sum over the blocks of $\pi \wedge \pi'$, the infimum of $\pi$ and $\pi'$ in the partition lattice.

We take up certain statistical and mathematical properties of permanental partition models. In particular, we consider conditions under which permanental models are exchangeable, consistent and, in the time-varying case, reversible. Reversible models are important in many scientific contexts of temporally evolving physical systems [22]. Many physical phenomena are, at least intuitively, reversible, and so models for such phenomena should reflect this property. Consistency (under subsampling) is important in statistical inference, and reflects a coherence among a collection of probability distributions that permits out-of-sample statistical inference. Consistency also results in beautiful mathematical theory involving limiting stochastic processes as the size of the system grows to infinity; see e.g. [6] for an overview of the mathematical theory underlying consistent systems of partition-valued processes and see [27] for a discussion of consistency in statistical models. Consistency under subsampling belies an assumed lack of interference among particles in a physical system. As such, consistency may not be appropriate in some applications and, in keeping, the permanental model is consistent only under special choices of $\mathbf{K}$.

### 1.1. The $\alpha$-permanent of a matrix.

Gibbs models for both time-invariant and time-varying partition structures have a history in diverse settings and here we focus primarily on *permanental partition models*. Specifically, we introduce both the *permanental partition distribution* and its related family of *permanental partition transition probabilities*, whose energy function entails the $\alpha$-permanent of some *similarity matrix*.

The *permanent* of any real-valued matrix $\mathbf{X} := (X_{ij},\ 1 \leq i, j \leq n)$ is defined by

$$\operatorname{per}\mathbf{X} := \sum_{\sigma\in\mathcal{S}_n}\prod_{j=1}^{n} X_{j,\sigma(j)}, \tag{3}$$

where $\mathcal{S}_n$ is the symmetric group acting on $[n] := \{1,\ldots,n\}$. In pure mathematics, the permanent arises in representation theory of the symmetric group as the immanant associated to the trivial representation $\rho \equiv 1$ of $\mathcal{S}_n$ [16, 31]. In probability theory and statistical physics, permanents and determinants appear in spatial point process models of bosons and fermions, respectively. In contrast to Poisson point processes, permanental and determinantal processes incorporate dependence among the points in an aggregated point pattern with clumping (permanental) or repellence (determinantal). Relevant to fermions is the Pauli exclusion principle, by which no two particles can occupy the same quantum state, and, consistent with this principle, any matrix with two or more rows identical has zero determinant; this principle does not apply to bosons, for which the permanent is an appropriate kernel, see [11, 19, 29, 35]. In permanental point processes, $\mathbf{X}$ plays the role of a *correlation matrix*–a higher value of $X_{ij}$ indicates correlation between measurements on particles labeled $i$ and $j$. In the permanental partition models we study, $\mathbf{X}$ plays the role of a *similarity matrix*–a higher value of $X_{ij}$ indicates that particles $i$ and $j$ are more likely to occupy the same internal state.

In a different context, Gyires [17] studied a family of discrete probability distributions related to the permanent; and, more recently, a generalization of the permanent, called the $\alpha$-permanent, has been connected to some well-known distributions in probability and statistics [37, 38]. For $\alpha \in \mathbb{R}$, the $\alpha$-permanent of $\mathbf{X}$ is defined by

$$\operatorname{per}_\alpha\mathbf{X} := \sum_{\sigma\in\mathcal{S}_n}\alpha^{\#\sigma}\prod_{j=1}^{n} X_{j,\sigma(j)}, \tag{4}$$

where $\#\sigma$ denotes the number of cycles of $\sigma \in \mathcal{S}_n$. Valiant [36] first showed the computational complexity of the permanent is #P-complete; and, since this seminal work, several algorithms have been proposed for its approximation, e.g. [21, 26, 28]. It has been recently conjectured [10] that, aside from $\alpha$ a negative integer, computation of the $\alpha$-permanent is at least as complex as the permanent.

1.2. **Permanental Gibbs models.** While the permanent has received substantial attention in the literature, the $\alpha$-permanent has been mostly neglected, except by a few authors. In this paper, we draw on ideas from statistical mechanics (Bolzmann-Gibbs distributions and permanental processes), applied probability (the Ewens sampling formula) and linear algebra (the $\alpha$-permanent) to derive a natural family of distributions and transition probabilities on partitions of a finite set $[n]$. Throughout the paper, we generically assume that particles $i = 1,\ldots,n$ occupy fixed positions in some abstract medium $\mathcal{X}$, and $\mathbf{X} := (X_{ij},\ 1 \leq i, j \leq n)$ defines a *similarity matrix* so that $X_{ij}$ quantifies the similarity (or interaction) between particles $i$ and $j$. Throughout the paper, we assume that $\mathbf{X}$ satisfies

- $X_{ii} = 1$ for all $1 \leq i \leq n$,
- $0 \leq X_{ij} \leq 1$ for $1 \leq i \neq j \leq n$, and
- $X_{ij} = X_{ji}$ for all $1 \leq i, j \leq n$.

For every $\alpha > 0$, the $\alpha$-permanent of any similarity matrix is strictly positive, as is the $\alpha$-permanent of $\mathbf{X}[b]$ for every subset $b \subseteq [n]$, where $\mathbf{X}[b] := (X_{ij},\ i, j \in b)$ is the submatrix of

$\mathbf{X}$ with rows and columns indexed by $b$. Therefore, we can use the $\alpha$-permanent to weight each partition of $[n]$, which determines a probability measure by normalization. Let $\pi$ be a partition of $[n]$ for which we write $b \in \pi$ to denote that $b$ is a block of $\pi$. Then, in the Boltzmann-Gibbs form, the permanental distribution can be expressed as

$$P_n^{\alpha,\beta,\mathbf{X}}(\pi) = \frac{a(\beta,\pi)}{Z_{\alpha,\beta,\mathbf{X}}} \exp\left\{\sum_{b\in\pi} \log \mathrm{per}_\alpha \mathbf{X}[b]\right\}. \tag{5}$$

In a similar way, we define the permanental transition probability from partition $\pi$ to $\pi'$ by

$$p_n^{\alpha,\beta,\mathbf{X}}(\pi,\pi') = \frac{a(\beta,\pi')}{Z_{\alpha,\beta,\mathbf{X}}(\pi)} \exp\left\{\sum_{b\in\pi\wedge\pi'} \log \mathrm{per}_\alpha \mathbf{X}[b]\right\}. \tag{6}$$

In both of these expressions, the additional factor $a(\cdot,\cdot)$ arises when considering Gibbs measures on the space of set partitions, and is consistent with previous literature on Gibbs partitions, e.g. Chapter 1.5 of [33].

1.3. **Computing the partition function.** In statistical mechanics, the partition function $Z_{\beta,H}$ in (1) plays an important role in understanding Gibbs measures in thermodynamic equilibrium. However, except in special cases, the partition function cannot be computed explicitly. A remarkable aspect of the permanental partition models is that the partition function can always be evaluated in closed form because of the Permanent Decomposition Theorem from [10] (reproduced as Theorem 3.1). In this case, (5) and (6) are given by

$$P_n^{\alpha,\beta,\mathbf{X}}(\pi) := \beta^{\downarrow\#\pi} \frac{\prod_{b\in\pi} \mathrm{per}_\alpha \mathbf{X}[b]}{\mathrm{per}_{\alpha\beta} \mathbf{X}} \quad \text{and} \tag{7}$$

$$p_n^{\alpha,\beta,\mathbf{X}}(\pi,\pi') := \beta^{\downarrow\#\pi'} \prod_{b\in\pi} \frac{\prod_{b'\in\pi'} \mathrm{per}_\alpha \mathbf{X}[b\cap b']}{\mathrm{per}_{\alpha\beta} \mathbf{X}[b]}, \tag{8}$$

respectively.

Knowledge of the partition function allows us to study this family of models in more detail than general Gibbs measures. For example, we observe an intimate connection between $P_n^{\alpha,\beta,\mathbf{X}}$ in (7) and the Pitman-Ewens distribution from population genetics [14, 33]; we discuss this connection further in Section 4. The permanental distributions can, therefore, be viewed as a three-parameter extension to the Pitman-Ewens two-parameter model. Analogously, the permanental transition probabilities (8) refine the self-similar cut-and-paste chains studied in [8, 9]; we discuss this further in Section 5.

1.4. **Summary of main theorems.** Far from a merely algebraic observation, the permanental partition families exhibit some fundamental modeling properties, adding to their practical appeal. Furthermore, any potential applications further emphasize the importance of understanding the $\alpha$-permanent, which has not been well-studied in the literature.

Before going through our main discussion, we highlight our main theorems and explain the structure of the paper. Our main results establish fundamental properties of both (7) and (8). Though we have not yet formally defined the parameter space of either (7) or (8), we assume, throughout this section, that $(\alpha,\beta,\mathbf{X})$ is a valid parameter choice, i.e. (7) and/or (8) determine a probability distribution on $\mathcal{P}_{[n]}$. We formally define the parameter space for these models in Definitions 4.6 and 5.8.

Note that, in Theorems 4.9 and 5.12, we say $\mathbf{X}$ corresponds to an infinite partition of $\mathbb{N}$ if $X_{ij} := \mathbb{I}\{i \sim j\}$ for some equivalence relation $\sim$ on $\mathbb{N}$.

1.4.1. *Gibbs partition structures.* Clearly, by (5), $P_n^{\alpha,\beta,\mathbf{X}}$ in (7) has the Gibbs form. Furthermore, for $\mathbf{X} = \mathbf{1}_n$, the matrix of all ones, we see that $P_n^{\alpha,\beta,\mathbf{1}_n}$ coincides with the Pitman-Ewens$(-\alpha, \alpha\beta)$-law on $\mathcal{P}_{[n]}$, discussed in detail in Section 4.1. In general, the Pitman-Ewens distribution characterizes the class of consistent canonical Gibbs ensembles on $(\mathcal{P}_{[n]},\ n \in \mathbb{N})$.

- **Theorem 4.8**: The permanental partition structure $P_n^{\alpha,\beta,\mathbf{X}}$ in (7) is exchangeable if and only if there is some $0 \leq d \leq 1$ such that $\mathbf{X}$ satisfies $X_{ij} = d$ for all $i \neq j$ and $X_{ii} = 1$ for all $i = 1, \ldots, n$.
- **Theorem 4.9**: Let $\mathbf{X} := (X_{ij},\ i, j \geq 1)$ be an infinite array corresponding to an infinite partition of $\mathbb{N}$ and, for each $n \in \mathbb{N}$, let $\mathbf{X}_n := (X_{ij},\ 1 \leq i, j \leq n)$ denote its leading $n \times n$ submatrix. Then the collection $(P_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ is consistent under subsampling.
- **Corollary 4.11**: Let $(P_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ be a collection of permanental partition structures such that, for each $n \in \mathbb{N}$, $P_n^{\alpha,\beta,\mathbf{X}_n}$ is a probability distribution on $\mathcal{P}_{[n]}$. Then $(P_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ is both exchangeable and consistent if and only if $\beta \in \mathbb{N}$ and, for either $d = 0$ or $d = 1$, $\mathbf{X}_n := (X_{ij},\ 1 \leq i, j \leq n)$, where $(X_{ij},\ i, j \geq 1)$ is an infinite array with entries $X_{ij} = d$ if $i \neq j$ and $X_{ii} = 1$ for all $i \in \mathbb{N}$.

1.4.2. *Markovian Gibbs partition structures.*

- **Theorem 5.9**: For all $(\alpha, \beta, \mathbf{X})$, the permanental Markovian Gibbs structure $p_n^{\alpha,\beta,\mathbf{X}}$ in (8) is reversible with respect to the permanental partition structure (7) with parameter $(\alpha\beta, \beta, \mathbf{X})$.
- **Theorem 5.10**: The permanental Markovian Gibbs structure $p_n^{\alpha,\beta,\mathbf{X}}$ in (8) is exchangeable if and only if there is some $0 \leq d \leq 1$ such that $\mathbf{X}$ satisfies $X_{ij} = d$ for all $i \neq j$ and $X_{ii} = 1$ for all $i = 1, \ldots, n$.
- **Theorem 5.12**: Let $\mathbf{X} := (X_{ij},\ i, j \geq 1)$ be an infinite array corresponding to an infinite partition of $\mathbb{N}$ and, for each $n \in \mathbb{N}$, let $\mathbf{X}_n := (X_{ij},\ i, j \geq 1)$ denote its leading $n \times n$ submatrix. Then the family $(p_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ is consistent under subsampling.
- **Corollary 5.13**: Let $(p_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ be a family of permanental Markovian Gibbs structures such that, for each $n \in \mathbb{N}$, $p_n^{\alpha,\beta,\mathbf{X}_n}$ is a transition probability on $\mathcal{P}_{[n]}$. Then $(p_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ is exchangeable and consistent if and only if $\beta \in \mathbb{N}$ and, for either $d = 0$ or $d = 1$, $\mathbf{X}_n := (X_{ij},\ 1 \leq i, j \leq n)$, where $(X_{ij},\ i, j \geq 1)$ is an infinite array with entries $X_{ij} = d$ if $i \neq j$ and $X_{ii} = 1$ for all $i \in \mathbb{N}$.

In addition to the above properties, the permanental partition models are equivariant with respect to the symmetric group, and we can construct an exchangeable model by mixing with respect to a random matrix $\mathbf{X}$ for which $\mathbf{X} =_{\mathcal{L}} \mathbf{X}^{\sigma}$ for all $\sigma \in \mathcal{S}_n$, where $=_{\mathcal{L}}$ denotes *equality in law*. We take these latter two properties up briefly in Section 6. Based on this summary, the permanental partition model has no *a priori* obvious modeling defects; that is, it is equivariant with respect to the natural action of the symmetric group by relabeling particles and, specific to the Markov chain setting, the permanental Markovian Gibbs structure is reversible. Moreover, we can derive conditions under which exchangeability and consistency are satisfied (both jointly and separately). Finally, far from an *ad hoc* choice based on the identity (19), the model extends the celebrated Pitman-Ewens family and, furthermore, the extra parameter **X** offers an intuitive interpretation in several applications.

In Section 2, we give some preliminaries for partitions and random partition models. In Section 3, we present some preliminary properties of the $\alpha$-permanent. In Section 4, we discuss the permanental family of distributions $P_n^{\alpha,\beta,\mathbf{X}}$ introduced above. In Section 5, we discuss the permanental family of transition probabilities. In Section 6, we briefly discuss the properties of equivariance and lack of interference in the context of the permanental partition model, as well as a related family of distributions on the space of $k$-colorings.

## 2. Preliminaries: random set partitions

2.1. **Partitions.** For $n \in \mathbb{N}$, a *partition* $\pi$ of $[n] := \{1, \dots, n\}$ is a collection $\{\pi_1, \dots, \pi_r\}$ of non-empty, non-overlapping subsets (blocks) such that $\bigcup_j \pi_j = [n]$. We write $\mathcal{P}_{[n]}$ to denote the collection of all partitions of $[n]$. Equivalently, we can define a partition of $[n]$ as an equivalence relation $\sim_\pi$, where

$$i \sim_\pi j \quad \Longleftrightarrow \quad i \text{ and } j \text{ are in the same block of } \pi.$$

From the definition of $\sim_\pi$, we may represent $\pi$ as an $n \times n$ matrix $(\pi_{ij},\ 1 \leq i, j \leq k)$ with $\pi_{ij} = 1$ if $i \sim_\pi j$ and $\pi_{ij} = 0$ otherwise. Throughout the paper, the equivalence of these three representations (as a collection of subsets, as an equivalence relation and as a matrix) is understood, and we need not specify to which representation we are referring.

For each $n \in \mathbb{N}$, $\mathcal{P}_{[n]}$ is partially ordered by $\leq$, called *sub-partition*. In particular, for $\pi, \pi' \in \mathcal{P}_{[n]}$, we define the infimum of $\pi$ and $\pi'$ by

$$\pi \wedge \pi' := \{b \cap b' :\ b \in \pi,\ b' \in \pi'\} \setminus \{\emptyset\}.$$

We then say that $\pi$ is a *sub-partition*, or *refinement*, of $\pi'$, written $\pi \leq \pi'$, if and only if $\pi = \pi \wedge \pi'$. We also write $\#\pi$ to denote the number of blocks of $\pi \in \mathcal{P}_{[n]}$.

Given $\pi \in \mathcal{P}_{[n]}$ and $m \leq n$, we define the *restriction* of $\pi$ to $[m]$ by

$$\mathbf{D}_{m,n}\, \pi = \pi_{|[m]} := \{b \cap [m],\ b \in \pi\} \setminus \{\emptyset\}, \tag{9}$$

the partition obtained by restricting each block of $\pi$ to $[m]$ and removing any empty sets. Under this restriction operation, the system $(\mathcal{P}_{[n]},\ n \in \mathbb{N})$ is projective and determines the limiting space $\mathcal{P}_\infty$ of partitions of $\mathbb{N}$. The projective structure of $(\mathcal{P}_{[n]},\ n \in \mathbb{N})$ is logically important in statistical applications and allows one to build processes on the abstract space $\mathcal{P}_\infty$ by specifying a consistent collection of finite-dimensional marginal distributions.

2.2. **Random partitions.** Random partitions and partition-valued processes arise in population genetics [14, 24, 25] to model evolutionary processes as well as in the physical sciences, e.g. [1], to model clustering of particles. In addition to these applications, the theory of exchangeable partitions is deeply connected to classical theory of stochastic processes, such as Brownian motion and Lévy processes, see [6, 33].

Specializing (1) to $\mathcal{P}_{[n]}$, we define a Gibbs measure on $\mathcal{P}_{[n]}$ by the form

$$P_n\{\Pi = \pi\} := \frac{a(\pi)}{Z_{a,H}} \exp\left\{\sum_{b \in \pi} H(b)\right\}, \quad \pi \in \mathcal{P}_{[n]}\,. \tag{10}$$

The additional factor $a(\pi)$ in (10) arises when we induce a Gibbs measure on $\mathcal{P}_{[n]}$ by projecting from a richer space. For example, if the internal states of a system are given explicitly by $x^1 \cdots x^n \in \mathbb{Z}^{[n]}$, a Gibbs measure on $\mathbb{Z}^{[n]}$ induces a measure on $\mathcal{P}_{[n]}$ through the relation

$$i \text{ and } j \text{ are in the same block of } \pi \quad \Longleftrightarrow \quad x^i = x^j.$$

In projecting to $\mathcal{P}_{[n]}$, the Gibbs form from (1) becomes (10). A special case of (10) is the so-called *canonical Gibbs ensemble* on $\mathcal{P}_{[n]}$ which, for sequences of non-negative coefficients $v_\bullet := (v_n,\ n \in \mathbb{N})$ and $w_\bullet := (w_n,\ n \in \mathbb{N})$, has the form

$$P_n(\pi; v_\bullet, w_\bullet) := \frac{v_{\#\pi}}{B_n(v_\bullet, w_\bullet)} \prod_{b \in \pi} w_{\#b}, \quad \pi \in \mathcal{P}_{[n]}\,. \tag{11}$$

In (11), $B_n(v_\bullet, w_\bullet)$ is the $n$th Bell polynomial associated to $(v_\bullet, w_\bullet)$. We call $P_n(\cdot; v_\bullet, w_\bullet)$ the $\text{Gibbs}_{[n]}(v_\bullet, w_\bullet)$ distribution on $\mathcal{P}_{[n]}$. We also call Gibbs distributions of the form (11) *mean-field*, meaning that the probability of any partition $\pi \in \mathcal{P}_{[n]}$ depends only on its block sizes. In the probability literature, mean-field models are called exchangeable and reflect invariance of $P_n(\cdot)$ under relabeling of the particles. In particular, for any permutation $\sigma : [n] \to [n]$, we define the image of $\pi$ by $\sigma$ by the relation $\sim_{\pi^\sigma}$, where

$$i \sim_{\pi^\sigma} j \quad \Longleftrightarrow \quad \sigma(i) \sim_\pi \sigma(j).$$

A distribution $P_n$ on $\mathcal{P}_{[n]}$ is called *exchangeable* if $P_n\{\Pi = \pi\} = P_n\{\Pi = \pi^\sigma\}$ for all permutations $\sigma : [n] \to [n]$. Clearly, a canonical Gibbs ensemble (11) is exchangeable for any choice of $(v_\bullet, w_\bullet)$.

When the weight sequences $v_\bullet$ and $w_\bullet$ are comprised of non-negative integers, the canonical $\text{Gibbs}_{[n]}(v_\bullet, w_\bullet)$ ensemble has a natural physical interpretation ([33], Chapter 1.5): any $\Pi \sim \text{Gibbs}_{[n]}(v_\bullet, w_\bullet)$ can be regarded as a random $(V \circ W)$-composite structure. By $(V \circ W)$-composite structure we mean that $\Pi$ is taken as a uniform draw from the set of all marked partitions of $[n]$, where each $\pi \in \mathcal{P}_{[n]}$ with $k$ blocks and block sizes $n_1 \geq \cdots \geq n_k > 0$ is assigned one of $v_k$ $V$-structures, and each block of $\pi$ of size $n_i$ is assigned one of $w_{n_i}$ $W$-structures, independently of other blocks. The distribution of $\Pi$ is then obtained by forgetting the $(V \circ W)$-structure and looking only at the induced partition of $[n]$.

2.3. **Integer partitions.** A partition of the integer $n \in \mathbb{N}$, which we shall call an *integer partition* of $n$, is a list $\mathbf{n} := (m_1, \ldots, m_n)$ of *multiplicities* for which $m_j \geq 0$, for all $j = 1, \ldots, n$, and $\sum_{j=1}^n j m_j = n$. We write $\mathcal{P}_n$ to denote the collection of integer partitions of $n$. Any $\pi \in \mathcal{P}_{[n]}$ induces an integer partition $\mathbf{n}(\pi)$ of $n \in \mathbb{N}$ through the map $\mathbf{n} : \mathcal{P}_{[n]} \to \mathcal{P}_n$, $\pi \mapsto \mathbf{n}(\pi) := (m_1, \ldots, m_n)$, where $m_j$ is the number of blocks of $\pi$ of size $j$ and $\sum_{j=1}^n m_j = \#\pi$.

Aside from its natural combinatorial and physical interpretations, a canonical $\text{Gibbs}_{[n]}(v_\bullet, w_\bullet)$ distribution (11) depends on $\pi \in \mathcal{P}_{[n]}$ only through $\mathbf{n}(\pi)$. To each $\mathbf{n} \in \mathcal{P}_n$, there are

$$\frac{n!}{\prod_{j=1}^n (j!)^{m_j} m_j!}$$

set partitions of $[n]$ in $\{\pi \in \mathcal{P}_{[n]} : \mathbf{n}(\pi) = \mathbf{n}\}$. Hence, given a probability distribution $P\{\mathbf{N} = \cdot\}$ on $\mathcal{P}_n$, we can generate a random partition of $[n]$ by first drawing its block sizes from $P\{\mathbf{N} = \cdot\}$ and, given $\mathbf{N} = \mathbf{n}$, we draw uniformly from $\{\pi \in \mathcal{P}_{[n]} : \mathbf{n}(\pi) = \mathbf{n}\}$. In some cases, as we see below, this induced distribution on $\mathcal{P}_{[n]}$ can be expressed as a canonical Gibbs ensemble.

**Example 2.1** (Ewens sampling formula)**.** *The Ewens sampling formula was first introduced by Ewens* [14] *as a probability distribution on integer partitions induced by allele frequencies in* *population genetics. For fixed $\theta > 0$, the Ewens sampling formula on $\mathcal{P}_n$ is given by*

$$P_n\{\mathbf{N} = \mathbf{n}\} = \frac{\theta^{\sum_{j=1}^n m_j}}{\theta^{\uparrow n}} \frac{n!}{\prod_{j=1}^n j^{m_j} m_j!}, \quad \mathbf{n} := (m_1, \ldots, m_n) \in \mathcal{P}_n . \tag{12}$$

*From the above discussion, the Ewens sampling formula induces the Ewens($\theta$) distribution on $\mathcal{P}_{[n]}$,*

$$P_n^{0,\theta}\{\Pi = \pi\} = \frac{\theta^{\#\pi}}{\theta^{\uparrow n}} \prod_{b\in\pi} (\#b - 1)!, \quad \pi \in \mathcal{P}_{[n]} .$$

*Note that the Ewens($\theta$) distribution is of canonical Gibbs type for $v_\bullet := \theta^\bullet$ and $w_\bullet := (\bullet - 1)!$.*

*The Pitman-Ewens distribution is a two-parameter extension of the Ewens distribution which, for suitable choices of $(\alpha, \theta)$, is defined by*

$$P_n^{\alpha,\theta}\{\Pi = \pi\} = \frac{(\theta/\alpha)^{\uparrow\#\pi}}{\theta^{\uparrow n}} \prod_{b\in\pi} -(-\alpha)^{\uparrow\#b}, \quad \pi \in \mathcal{P}_{[n]},$$

*where $\alpha^{\uparrow j} := \alpha(\alpha + 1)\cdots(\alpha + j - 1)$. Note that the Pitman-Ewens distribution is a canonical Gibbs ensemble with $v_\bullet := (\theta/\alpha)^{\uparrow\bullet}$ and $w_\bullet := -(-\alpha)^{\uparrow\bullet}$. We discuss the Pitman-Ewens family in more detail in Section 4.1.*

2.4. **Markovian Gibbs partition structures.** Following the discussion in Section 2.2, we introduce *Markovian Gibbs structures* for time-varying Gibbs partitions. A Markovian Gibbs structure is a transition probability measure on $\mathcal{P}_{[n]}$ of the form

$$p_n(\pi, \pi') := \frac{a(\pi')}{Z_{a,H}(\pi)} \exp\left\{\sum_{b\in\pi} \sum_{b'\in\pi'} H(b, b')\right\}, \quad \pi, \pi' \in \mathcal{P}_{[n]}, \tag{13}$$

where $H : 2^{[n]} \times 2^{[n]} \to \mathbb{R}$ is the *joint energy function*. The form in (13) is natural as, in some cases, it admits an interpretation of independent partitioning of each of the blocks of $\pi$ and, subsequently, merging blocks together to obtain $\pi'$. The self-similar cut-and-paste chain is an example of a Markovian Gibbs structure with this interpretation.

**Example 2.2** (Self-similar cut-and-paste chain)**.** *For $k \geq 1$, let $\mathcal{P}_{[n]:k}$ denote the collection of partitions of $[n]$ having at most $k$ blocks. For $\alpha > 0$, Crane [8] introduced a family of Markov chains on $\mathcal{P}_{[n]:k}$ with transition probabilities*

$$p_n(\pi, \pi') = k^{\downarrow\#\pi'} \prod_{b\in\pi} \frac{\prod_{b'\in\pi'} (\alpha/k)^{\uparrow\#(b\cap b')}}{\alpha^{\uparrow\#b}}, \quad \pi, \pi' \in \mathcal{P}_{[n]:k}, \tag{14}$$

*where $k^{\downarrow j} := k(k - 1)\cdots(k - j + 1)$. Note that (14) has the form of (13) for $a(\pi') := k^{\downarrow\#\pi'}$, $H(b, b') := (\alpha/k)^{\uparrow\#(b\cap b')}$ and $Z_{a,H}(\pi) := \prod_{b\in\pi} \alpha^{\uparrow\#b}$. Note also that (14) can be expressed as*

$$p_n(\pi, \pi') = \frac{k^{\downarrow\#\pi'}}{\prod_{b\in\pi} k^{\downarrow\pi'_{|b}}} \prod_{b\in\pi} \frac{k^{\downarrow\#\pi'_{|b}}}{\alpha^{\uparrow\#b}} \prod_{b'\in\pi'} (\alpha/k)^{\uparrow\#(b\cap b')}, \quad \pi, \pi' \in \mathcal{P}_{[n]:k},$$

*which has a straightforward interpretation. First, we independently partition each block of $\pi$ according to the Pitman-Ewens$(-\alpha/k, \alpha)$ distribution ($P_n^{-\alpha/k,\alpha}$ from Example 2.1); we then label each of the sub-blocks within each block distinctly in $[k]$ by drawing uniformly without replacement; finally, we put all sub-blocks assigned the same label into a single block in $\pi'$ and remove labels.*

Motivated by the above interpretation of the cut-and-paste transition probabilities and the cut-and-paste representation of exchangeable Feller chains in [9], we focus on joint energy functions depending on the pair of subsets $(b, b')$ only through the intersection $b \cap b'$. Under this convention, we define a *canonical Markovian Gibbs structure*, or *mean-field Markovian Gibbs structure*, on $\mathcal{P}_{[n]}$ by the form

$$p_n(\pi, \pi'; v_\bullet, w_\bullet) := \frac{v_{\#\pi'}}{B_n(v_\bullet, w_\bullet, \pi)} \prod_{b \in \pi \wedge \pi'} w_{\#b}, \quad \pi, \pi' \in \mathcal{P}_{[n]}\,. \tag{15}$$

In the case where both $v_\bullet$ and $w_\bullet$ are integer-valued, (15) has an analogous interpretation, in terms of $(V \circ W)$-composite structures, as in the one-dimensional case above. We call a Markovian Gibbs structure $p_n(\cdot, \cdot)$ *exchangeable* if it is invariant under joint relabeling of the argument $(\pi, \pi')$, that is, $p_n(\pi, \pi') = p_n(\pi^\sigma, \pi'^\sigma)$ for all permutations $\sigma : [n] \to [n]$, for all $\pi, \pi' \in \mathcal{P}_{[n]}$. Because, for any $\sigma : [n] \to [n]$, $\pi^\sigma \wedge \pi'^\sigma = (\pi \wedge \pi')^\sigma$, it is clear that any canonical Markovian Gibbs ensemble (15) is exchangeable.

2.5. **Model properties.** We consider several fundamental properties of partition models in this paper. We have already mentioned exchangeability in Sections 2.2 and 2.4. Later, in Section 6, we discuss the property of *equivariance*, which pertains to models that associate *covariates* to the particles of the system. In the permanental models we study, the covariates are the positions of particles in space, resulting in a parameter $\mathbf{X}$ that is not invariant under relabeling. However, if we relabel the particles and the similarity matrix jointly by the same permutation, the model is unchanged. Other properties include consistency under subsampling and reversibility, which we now discuss.

2.5.1. *Consistency under subsampling.* A consideration that arises in statistical inference, and elsewhere when studying incompletely observed systems, is *consistency under subsampling*. A family of measures $\{P_n,\ n \in \mathbb{N}\}$, where $P_n$ is a probability measure on $\mathcal{P}_{[n]}$ for every $n \in \mathbb{N}$, is *consistent* if, for every $m \leq n$,

$$P_m(\pi) = P_n(\mathbf{D}_{m,n}^{-1}(\pi)), \quad \text{for all } \pi \in \mathcal{P}_{[n]}\,. \tag{16}$$

Particularly in Bayesian posterior predictive inference, a consistent family $\{P_n,\ n \in \mathbb{N}\}$ determines a predictive statistical model in that, given an observation $\pi \in \mathcal{P}_{[m]}$, the model gives a set of conditional probability distributions on the unobserved part of the population by

$$P_{n|m}(\pi' \,|\, \pi) := \begin{cases} P_n(\pi')/P_m(\pi), & \pi = \mathbf{D}_{m,n}\,\pi' \\ 0, & \text{otherwise.} \end{cases}$$

Consistent collections on $(\mathcal{P}_{[n]},\ n \in \mathbb{N})$ play a particularly important role in Bayesian classification and clustering [30]. In classical probability theory, consistent collections of probability measures determine a unique probability measure on some limiting space, in this case the space $\mathcal{P}_\infty$ of partitions of $\mathbb{N}$. It is known that Gibbs distributions are not consistent in general, see e.g. [34] for an illustration in the case of the Exponential Random Graph Model.

For a Markov chain, consistency under subsampling corresponds to the condition that, for every $m \leq n$, the restriction to $\mathcal{P}_{[m]}$ of a $\mathcal{P}_{[n]}$-valued Markov chain is, itself, a Markov chain. Since the projection operation $\mathbf{D}_{m,n} : \mathcal{P}_{[n]} \to \mathcal{P}_{[m]}$ is many-to-one, it does not immediately follow that the image of a Markov chain by restriction will maintain the Markov property. By a theorem of Burke and Rosenblatt [7], it is necessary and sufficient that, for every $m \leq n$,

$$P_m(\pi, \pi') = P_n(\pi^*, \mathbf{D}_{m,n}^{-1}(\pi')) \quad \text{for all } \pi^* \in \mathbf{D}_{m,n}^{-1}(\pi), \tag{17}$$

for every $\pi, \pi' \in \mathcal{P}_{[m]}$. Under this condition, there exists a transition probability measure on $\mathcal{P}_\infty$ and, hence, a Markov chain on $\mathcal{P}_\infty$ with that transition probability measure.

2.5.2. *Reversibility.* A further property, relevant in the time-varying setting, is *reversibility*, whereby the law governing the evolution of a physical system is invariant under time-reversal. Given a Markov transition probability $p_n(\cdot,\cdot)$ on $\mathcal{P}_{[n]}$ and a measure $\mu_n(\cdot)$ on $\mathcal{P}_{[n]}$, $p_n$ is reversible with respect to $\mu_n$ if it satisfies the *detailed-balance equation*

$$\mu_n(\pi)p_n(\pi,\pi') = \mu_n(\pi')p_n(\pi',\pi) \quad \text{for all } \pi,\pi' \in \mathcal{P}_{[n]}\,. \tag{18}$$

If (18) holds for a probability measure $\mu_n$, then $\mu_n$ is the stationary distribution for any Markov chain governed by $p_n(\cdot,\cdot)$.

## 3. Preliminaries: the $\alpha$-permanent

Throughout this section, let $\mathbf{X}$ be an $\mathbb{R}$-valued $n \times n$ matrix and, for $\alpha \in \mathbb{R}$, recall the definition of the $\alpha$-permanent in (4). We note that some authors, e.g. [19], prefer to define the $\alpha$-determinant $\det_\alpha \mathbf{X}$ by

$$\det_\alpha \mathbf{X} := \sum_{\sigma\in\mathcal{S}_n} \alpha^{n-\#\sigma} \prod_{j=1}^n X_{j,\sigma(j)},$$

which relates to (4) through $\mathrm{per}_\alpha \mathbf{X} = \alpha^n \det_{1/\alpha} \mathbf{X}$.

For all $\alpha \geq 0$, any $\mathbf{X}$ with $X_{ij} \geq 0$ for all $1 \leq i,j \leq n$ has a non-negative $\alpha$-permanent and, moreover, the $\alpha$-permanent of every submatrix of $\mathbf{X}$ is non-negative. Non-negativity of $\mathrm{per}_\alpha \mathbf{X}[b]$ for all $b \subseteq [n]$ plays the same role as positive definiteness in the related study of determinantal distributions on subsets of $[n]$. In particular, when $\mathbf{X}$ is positive definite, $\det \mathbf{X}[b] \geq 0$ for all $b \subseteq [n]$ because all leading minors of a positive definite matrix are positive and the determinant is invariant under conjugation by a permutation matrix. Furthermore, the normalizing constant is given by $\sum_{b\subseteq[n]} \det \mathbf{X}[b] = \det(I_n + \mathbf{X})$, where $I_n$ is the $n \times n$ identity matrix (see e.g. Theorem 1.2 of [10]).

**Theorem 3.1** (Theorem 1.1 in [10])**.** *Let $\alpha, \beta \in \mathbb{C}$ and $M$ be any $n \times n$ $\mathbb{C}$-valued matrix. Then the $\alpha$-permanent satisfies the identity*

$$\mathrm{per}_{\alpha\beta} M = \sum_{\pi\in\mathcal{P}_{[n]}} \beta^{\downarrow\#\pi}\, \mathrm{per}_\alpha(M \cdot \pi), \tag{19}$$

*where $\pi := (\pi_{ij})$ is regarded as a 0-1 valued matrix representing a partition of $[n]$, and $M \cdot \pi := (M_{ij}\pi_{ij},\ 1 \leq i,j \leq n)$ is the Hadamard product of $M$ and $\pi$. In particular,*

$$\mathrm{per}_\alpha(M \cdot \pi) := \prod_{b\in\pi} \mathrm{per}_\alpha M[b].$$

From a purely algebraic perspective, (19) is interesting because it gives an explicit relationship between the $\alpha$-permanent and the determinant:

$$\mathrm{per}_\alpha \mathbf{X} = (-1)^n \sum_{\pi\in\mathcal{P}_{[n]}} (-\alpha)^{\downarrow\#\pi} \det(\mathbf{X} \cdot \pi).$$

From a statistical mechanics and probabilistic perspective, (19) gives an explicit formula for the normalizing constant of a family of probability distributions on set partitions: provided each term on its right-hand side is non-negative and $\mathrm{per}_{\alpha\beta} \mathbf{X} > 0$, (19) determines a probability distribution $P_n^{\alpha,\beta,\mathbf{X}}$ on $\mathcal{P}_{[n]}$, as given in (7). We call $P_n^{\alpha,\beta,\mathbf{X}}$ the *permanental partition* *distribution*, or *permanental partition structure*, with parameter $(\alpha, \beta, \mathbf{X})$. In general, many combinations of $\alpha, \beta$ and $\mathbf{X}$ determine a probability distribution on $\mathcal{P}_{[n]}$ through (7); however, in the coming sections, we focus on a parameterization that admits a straightforward interpretation. Similarly, we can define a transition probability on $\mathcal{P}_{[n]}$ as in (8).

## 4. Permanental partition structures

4.1. **The Pitman-Ewens model.** The Pitman-Ewens distribution is, perhaps, the most well-known and well-studied family of distributions on set partitions. In surprising ways, the Pitman-Ewens distribution arises in diverse applied and theoretical problems involving exchangeable random partitions. Here we show its close connection to the $\alpha$-permanent, which connects the Pitman-Ewens family to the permanental partition structures in (7).

For $\theta > 0$ and $n \in \mathbb{N}$, let us define

$$P_n^{0,\theta}(\pi) := \frac{\theta^{\#\pi}}{\theta^{\uparrow n}} \prod_{b \in \pi} (\#b - 1)!, \quad \pi \in \mathcal{P}_{[n]}, \tag{20}$$

where $\theta^{\uparrow n} := \theta(\theta+1)\cdots(\theta+n-1)$. The distribution $P_n^{0,\theta}$ is called the Ewens distribution with parameter $\theta$ (as in Example 2.1). Viewed as a model in population genetics, the parameter $\theta$ relates to the mutation rate, as higher values of $\theta$ correspond to a larger expected number of blocks and, hence, more mutation.

There are numerous connections between (20) and topics in probability theory, number theory and combinatorics. In probability theory alone, there are connections between the Ewens distribution and Brownian motion, Lévy processes and the Poisson-Dirichlet distribution. See [6, 33] for recent surveys of this work. The next two examples illustrate the diverse scenarios in which (20) arises.

**Example 4.1** (Ewens distribution in statistics)**.** *In statistics, the Ewens distribution arises, at least implicitly, in a species sampling context; see e.g.* [13, 15]*. In this setting, let $X_1, X_2, \ldots$ be independent Poisson random variables so that $X_j$ has mean $\theta/j$, for $\theta > 0$. For each $j \geq 1$, we interpret $X_j$ as the number of species appearing $j$ times in a sample of $n$ specimens. The conditional distribution of $X_1, \ldots, X_n$, given $\sum_j jX_j = n$, is easily computed by*

$$\mathbb{P}\left((X_1, \ldots, X_n) = (x_1, \ldots, x_n) \mid \sum_j jx_j = n\right) \propto e^{-\theta \sum_j 1/j} \frac{\theta^{\sum_j x_j}}{\prod_j j^{x_j} x_j!},$$

*which normalizes to the Ewens sampling formula* (12) *on integer partitions of $n$; see Example 2.1. The Ewens sampling formula with parameter $\theta$ gives the distribution of the block sizes of $\Pi \sim$ Ewens$(\theta)$, from which the mean number of blocks of $\Pi$ is readily observed to be*

$$\mathbb{E} \sum_{j=1}^{n} X_j = \theta \sum_{j=1}^{n} 1/j \sim \theta \log n.$$

**Example 4.2** (Ewens partitions in number theory, see [12])**.** *For a number theoretic connection, consider drawing a uniform random integer $N$ from $[n]$. For each $i = 1, \ldots, n$, let $F_i$ be the ith largest prime factor of $N$, where we put $F_i \equiv 1$ if $i$ exceeds the number of prime factors of $N$. Then the logarithmically transformed sequence* $(\log n)^{-1}(L_1, L_2, \ldots)$*, where $L_i = \log F_i$, converges in law to the Poisson-Dirichlet distribution with parameter* $(0, 1)$ [2]*. In general, the Ewens$(\theta)$ distribution in* (20) *corresponds to the Poisson-Dirichlet distribution with parameter* $(0, \theta)$ *through Kingman's paintbox correspondence* [23].

In addition to the previous two examples, the Ewens distribution is observed in other diverse mathematical and scientific contexts. In this paper, we highlight a new connection between permanental partition structures and the two-parameter Pitman-Ewens law. For $(\alpha, \theta)$ satisfying either

- $\alpha < 0$ and $\theta = -k\alpha$ for some $k \in \mathbb{N}$ or
- $0 \leq \alpha \leq 1$ and $\theta > -\alpha$,

the Pitman-Ewens distribution with parameter $(\alpha, \theta)$ is given by

$$P_n^{\alpha,\theta}(\pi) = \frac{(\theta/\alpha)^{\uparrow\#\pi}}{\theta^{\uparrow n}} \prod_{b\in\pi} -(-\alpha)^{\uparrow\#b}, \quad \pi \in \mathcal{P}_{[n]}\,. \tag{21}$$

For $(\alpha, \theta)$ in the parameter range of the Pitman-Ewens model, the collection $(P_n^{\alpha,\theta},\ n \in \mathbb{N})$ is both exchangeable and consistent, and, therefore, determines a unique probability measure $P^{\alpha,\theta}$ on $\mathcal{P}_\infty$.

4.2. **Gibbs partitions.** In specializing the Boltzmann-Gibbs measure (1) to the space of set partitions, we recall that Gibbs distributions on $\mathcal{P}_{[n]}$ have the form

$$P_n(\pi) := \frac{a(\pi)}{Z_{a,H}} \exp\left\{\sum_{b\in\pi} H(b)\right\}, \quad \pi \in \mathcal{P}_{[n]}, \tag{22}$$

for functions $a : \mathcal{P}_{[n]} \to \mathbb{R}$ and $H : 2^{[n]} \to \mathbb{R}$. Immediately, comparing (21) and (22), we see that the Pitman-Ewens$(\alpha, \theta)$ distribution is a Gibbs partition measure with $a(\pi) := (\theta/\alpha)^{\uparrow\#\pi}$, $H(b) := \log[-(-\alpha)^{\uparrow\#b}]$ and $Z_{a,H} := \theta^{\uparrow n}$. In the special case $\alpha = 0$, the Pitman-Ewens$(\alpha, \theta)$-law reduces to the Ewens$(\theta)$-law (20). Furthermore, the Pitman-Ewens$(\alpha, \theta)$-law (21) is a canonical Gibbs$_{[n]}(v_\bullet, w_\bullet)$ measure (11) with $v_\bullet := (\theta/\alpha)^{\uparrow\bullet}$ and $w_\bullet := -(-\alpha)^{\uparrow\bullet}$. The Ewens$(\theta)$ distribution is a Gibbs$_{[n]}(v_\bullet, w_\bullet)$ measure with $v_\bullet = 1^\bullet$ and $w_\bullet := (\bullet - 1)!$. In the following theorem, we recall the characterization of the Pitman-Ewens$(\alpha, \theta)$ distribution as the unique exchangeable and consistent Gibbs model on $\mathcal{P}_{[n]}$.

**Theorem 4.3.** *For weight sequences $v_\bullet, w_\bullet$, let $(P_n(\cdot; v_\bullet, w_\bullet),\ n \in \mathbb{N})$ be a family of canonical Gibbs measures. Then $(P_n(\cdot; v_\bullet, w_\bullet),\ n \in \mathbb{N})$ is consistent if and only if there exists $(\alpha, \theta)$ in the parameter space of the Pitman-Ewens model such that $v_\bullet = (\theta/\alpha)^{\uparrow\bullet}$ and $w_\bullet = -(-\alpha)^{\uparrow\bullet}$.*

*Proof.* Regard $P_n^{v_\bullet,w_\bullet}$ as a generic Gibbs$_{[n]}(v_\bullet, w_\bullet)$ distribution with weights $v_\bullet, w_\bullet$:

$$P_n^{v_\bullet,w_\bullet}(\pi) := \frac{v_{\#\pi}}{Z_n} \prod_{b\in\pi} w_{\#b}, \quad \pi \in \mathcal{P}_{[n]}\,.$$

Consistency of the family $(P_n^{v_\bullet,w_\bullet},\ n \in \mathbb{N})$ requires that, for every $n \in \mathbb{N}$,

$$P_n^{v_\bullet,w_\bullet}(\pi) = \sum_{\pi' \in \mathbf{D}^{-1}_{n,n+1}(\pi)} P_{n+1}^{v_\bullet,w_\bullet}(\pi') \quad \text{for all } \pi \in \mathcal{P}_{[n]}\,.$$

In the case of the canonical Gibbs$_{[n]}(v_\bullet, w_\bullet)$ distribution (11), consistency reduces to

$$\begin{aligned}
\frac{v_{\#\pi}}{Z_n} \prod_{b\in\pi} w_{\#b} &= \sum_{\pi' \in \mathbf{D}^{-1}_{n,n+1}(\pi)} \frac{v_{\#\pi'}}{Z_{n+1}} \prod_{b\in\pi'} w_{\#b} \\
&= \frac{1}{Z_{n+1}} \left[ v_{\#\pi+1} w_1 \prod_{b\in\pi} w_{\#b} + v_{\#\pi} \prod_{b\in\pi} w_{\#b} \sum_{b\in\pi} W_{\#b} \right],
\end{aligned}$$

where $W_i := w_{i+1}/w_i$ for $i \geq 1$. Now, suppose $\#\pi = k \geq 1$, then the above identity is identical to

$$\frac{Z_{n+1}}{Z_n} = V_k + \sum_{b\in\pi} W_{\#b},$$

for all $k \geq 1$, where $V_k := v_{k+1}/v_k$. Since the left-hand side does not depend on $\pi$, we must have, for every $i, j \geq 1$,

$$W_i + W_j = W_{i-1} + W_{j+1},$$

and, hence, $W_i - W_{i-1} = W_{j+1} - W_j$ for all $i, j \geq 1$. Therefore, $W_i$ is of the form $W_i = \alpha + \beta i$ for real constants $\alpha$ and $\beta$. Thus,

$$\frac{Z_{n+1}}{Z_n} = V_k + \alpha k + \beta n,$$

and so $V_k + \alpha k$ is constant in $k$. We, therefore, may write $V_k = \theta - \alpha k$, for some constant $\theta$ such that $V_k \geq 0$ for all $k \geq 1$ and $V_k > 0$ for $k = 1$. Hence,

$$v_k := \prod_{j=0}^{k-1} V_j = (\theta - \alpha(k-1))\cdots\theta = \begin{cases} \alpha^k\,(\theta/\alpha)^{\downarrow k}, & \alpha \neq 0 \\ \theta^k, & \alpha = 0. \end{cases}$$

For $0 < \alpha < \infty$, $v_k \geq 0$ for all $k \geq 1$ if and only if $\theta/\alpha$ is a positive integer. Furthermore, for each $j \geq 1$,

$$w_j = w_1 \prod_{i=1}^{j-1} W_i = \beta(\alpha+\beta)\cdots(\alpha+(j-1)\beta) = \begin{cases} \beta^j\,(\alpha/\beta+1)^{\uparrow(j-1)}, & \alpha \neq 0 \\ (j-1)!, & \alpha = 0, \end{cases}$$

where we have written $w_1 = \beta$. By the form of (11), we can force $\beta = w_1 = 1$ so that $w_j = (\alpha+1)^{\uparrow(j-1)}$ for $\alpha \neq 0$. We must have $\alpha + 1 \geq 0$ and, hence, $\alpha \geq -1$. For $-1 \leq \alpha \leq 0$, the condition $v_1 > 0$ forces $\theta > \alpha$. This completes the proof. □

In addition to Theorem 4.3, the Ewens($\theta$) distribution is of exponential family type with canonical sufficient statistic $\#\pi$ and natural parameter $\log\theta$. These observations point to why the Ewens distribution appears so widely in applications involving partitions. Not only are exponential family models widely used in statistical applications, but exchangeability and consistency are also important to the logical structure of statistical models [27]. Above, we have also recounted how the Ewens and Pitman-Ewens families arise out of purely mathematical concerns (Example 4.2). In the next section, we point out another appearance of the Pitman-Ewens family that relates to the permanental partition distribution (7).

4.3. **Permanental partition structures.** In (7), the permanental partition structure with parameter $(\alpha, \beta, \mathbf{X})$ is defined by

$$P_n^{\alpha,\beta,\mathbf{X}}(\pi) := \beta^{\downarrow\#\pi}\frac{\mathrm{per}_\alpha(\mathbf{X}\cdot\pi)}{\mathrm{per}_{\alpha\beta}\mathbf{X}}, \quad \pi \in \mathcal{P}_{[n]}, \tag{23}$$

for some choice of $\alpha, \beta \in \mathbb{R}$ and $\mathbf{X} \in \mathbb{R}^{n\times n}$. By identity (19), (23) is a probability distribution on $\mathcal{P}_{[n]}$, provided (23) is non-negative for every $\pi \in \mathcal{P}_{[n]}$ and $\mathrm{per}_{\alpha\beta}\mathbf{X} > 0$.

In comparision to (22), (23) is of Gibbs-type with $a(\pi) := \beta^{\downarrow\#\pi}$, $H(b) := \log\mathrm{per}_\alpha\mathbf{X}[b]$ and $Z_{\alpha,\beta,\mathbf{X}} := \mathrm{per}_{\alpha\beta}\mathbf{X}$. Writing $\mathbf{1}_n$ to denote the $n \times n$ matrix of all ones, we observe that

$\mathrm{per}_\alpha \mathbf{1}_n = \alpha^{\uparrow n}$; hence, for $\mathbf{X} = \mathbf{1}_n$ and an appropriate choice of $(\alpha, \beta)$, (23) reduces to

$$P_n^{\alpha,\beta,\mathbf{1}_n}(\pi) = \beta^{\downarrow \#\pi} \frac{\prod_{b \in \pi} \alpha^{\uparrow \#b}}{(\alpha\beta)^{\uparrow n}}, \quad \pi \in \mathcal{P}_{[n]},$$

the Pitman-Ewens$(-\alpha, \alpha\beta)$ distribution in (21). This observation highlights yet another appearance of the Pitman-Ewens distribution out of a seemingly unrelated algebraic observation.

**Proposition 4.4.** *For $n \in \mathbb{N}$, let $\alpha > 0$, $\beta \in [n] \cup [n, \infty)$ and $\mathbf{X}$ be an $n \times n$ $\mathbb{R}_+$-valued matrix such that $\mathrm{per}_{\alpha\beta} \mathbf{X} > 0$. Then* (23) *is a probability distribution on $\mathcal{P}_{[n]}$.*

*Proof.* If $\beta \in [n] \cup [n, \infty)$, then $\beta^{\downarrow \#\pi} \geq 0$ for all $\pi \in \mathcal{P}_{[n]}$. By (19) and the assumption $\mathrm{per}_{\alpha\beta} \mathbf{X} > 0$, (23) is a probability distribution on $\mathcal{P}_{[n]}$. □

For the rest of the paper, we always assume $\alpha > 0$ and $\beta \in [n] \cup [n, \infty)$ when discussing the permanental distribution (23) on $\mathcal{P}_{[n]}$. However, by our next proposition, the family $\{P_n^{\alpha,\beta,\mathbf{X}}\}_{\alpha,\beta,\mathbf{X}}$ of distributions on $\mathcal{P}_{[n]}$ is not identifiable when $\mathbf{X}$ is allowed to vary over the entire space of $\mathbb{R}_+$-valued matrices. For example, for fixed $\alpha, \beta$, the matrices $\mathbf{X}$ and $a\mathbf{X}$, where $a > 0$, determine the same probability distribution through (23), prompting our restriction of $\mathbf{X}$ to the space of *similarity matrices*. In particular, we call an $\mathbb{R}_+$-valued matrix $\mathbf{X} := (X_{ij}, 1 \leq i, j \leq n)$ a *similarity matrix* if

(i) $X_{ii} = 1$ for all $i = 1, \ldots, n$,
(ii) $0 \leq X_{ij} \leq 1$ for all $1 \leq i \neq j \leq n$, and
(iii) $X_{ij} = X_{ji}$ for all $1 \leq i, j \leq n$.

Although there is, in general, no harm in allowing off-diagonal values to exceed diagonal values, the interpretation of $\mathbf{X}$ as a measurement of pairwise similarity suggests that $X_{ij} \leq X_{ii} \wedge X_{jj}$ for all $i$ and $j$. Symmetry of $\mathbf{X}$ is also needed for identifiability. To be clear, only (i) and (iii) are necessary to make the model identifiable; item (ii) arises out of modeling concerns and not mathematical necessity.

**Proposition 4.5.** *Let $\mathbf{X}$ be any $n \times n$ non-negative matrix with strictly positive diagonal entries $X_{11}, \ldots, X_{nn}$ and define $\mathbf{X}'$ from $\mathbf{X}$ by*

$$X'_{ij} := X_{ij}/X_{ii}, \quad 1 \leq i, j \leq k. \tag{24}$$

*Let $\alpha, \beta$ satisfy the conditions of Proposition 4.4. Then the triples $(\alpha, \beta, \mathbf{X})$ and $(\alpha, \beta, \mathbf{X}')$ determine the same probability law on $\mathcal{P}_{[n]}$ through* (23).

*Proof.* For a generic matrix $M := (M_{ij}, 1 \leq i, j \leq n)$ and constants $a_1, \ldots, a_n > 0$, let $M^a := (M^a_{ij}, 1 \leq i, j \leq n)$ denote the matrix with values

$$M^a_{ij} := a_i M_{ij}, \quad 1 \leq i, j \leq k.$$

Then, for $\alpha \in \mathbb{R}$, $\mathrm{per}_\alpha M^a := \mathrm{per}_\alpha(M) \prod_{j=1}^n a_j$. Therefore, for $\mathbf{X}$ defined in (24), we have

$$P_n^{\alpha,\beta,\mathbf{X}}(\pi) = \beta^{\downarrow \#\pi} \frac{\mathrm{per}_\alpha(\mathbf{X} \cdot \pi)}{\mathrm{per}_{\alpha\beta} \mathbf{X}} = \beta^{\downarrow \#\pi} \frac{\mathrm{per}_\alpha(\mathbf{X}' \cdot \pi) \prod_{j=1}^n X_{jj}}{\mathrm{per}_{\alpha\beta}(\mathbf{X}') \prod_{j=1}^n X_{jj}} = P_n^{\alpha,\beta,\mathbf{X}'}(\pi),$$

for every $\pi \in \mathcal{P}_{[n]}$. □

**Definition 4.6** (Permanental partition distribution). *For* $n \in \mathbb{N}$, $\alpha > 0$, $\beta \in [n] \cup [n, \infty)$ *and any similarity matrix* $\mathbf{X}$, *we call* $P_n^{\alpha,\beta,\mathbf{X}}$ *in* (23) *the* $\textsc{Per}_n(\alpha, \beta, \mathbf{X})$*-partition structure, or* $\textsc{Per}_n(\alpha, \beta, \mathbf{X})$ *distribution, with* diversity $\alpha$, mixing coefficient $\beta$ *and* similarity matrix $\mathbf{X}$. *We call* $\{P_n^{\alpha,\beta,\mathbf{X}}\}$ *the permanental family of distributions on* $\mathcal{P}_{[n]}$.

4.3.1. *Parameterizing the* $\textsc{Per}_n(\alpha, \beta, \mathbf{X})$*-family.* There are some natural sub-parameter choices for $\mathbf{X}$ that may be appropriate in specialized settings. Per our discussion in Section 1, we interpret $\mathbf{X}$ as a parameterization of the medium, or environment, in which particles are suspended. For example, we refer to the case $\mathbf{X} = \mathbf{1}_n$ as a *uniformly homogeneous medium*, which reflects the assumption that every particle, regardless of its location, interacts with every other particle (including itself) in the same way. We have seen that the permanental partition model in a uniformly homogeneous medium corresponds to the Pitman-Ewens model.

We can generalize the uniformly homogeneous model to a *homogeneous medium*, in which case $\mathbf{X}$ corresponds to some symmetrically homogeneous matrix, $\mathbf{H}[1, d; n]$, $0 \le d \le 1$, where

$$\mathbf{H}[1, d; n] := \begin{pmatrix} 1 & d & \cdots & d \\ d & 1 & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & 1 \end{pmatrix}. \tag{25}$$

The special case $d = 0$ corresponds to $\mathbf{X} = I_n$, the $n \times n$ identity matrix, which we refer to as a *discrete medium*. In a discrete medium, each particle interacts only with itself.

Not covered by the above homogeneous media cases is the case of *non-overlapping uniformly homogeneous sub-populations*, in which we can let $\mathbf{X}$ correspond to a partition $\pi$ of $[n]$ by the specification

$$X_{ij} = \begin{cases} 1, & i \sim_\pi j \\ 0, & \text{otherwise.} \end{cases}$$

In this case, each block of the partition acts as a uniformly homogeneous medium over a sub-population, but particles in different sub-populations do not interact. By a further generalization to the non-overlapping sub-population case, we can define, for each $b \in \pi$, a parameter $0 \le d_b \le 1$ so that, within each sub-population, particles interact as if suspended in a homogeneous medium described by $\mathbf{H}[1, d_b; \#b]$, but particles in disjoint blocks do not interact. As we progress, we discuss these parameter choices in some detail.

Another choice for $\mathbf{X}$, natural in phylogenetic modeling, is the case where $\mathbf{X}$ corresponds to a rooted-tree matrix. Within our parameterization of $\mathbf{X}$, a rooted-tree matrix is a similarity matrix satisfying the additional *three-point condition*

$$X_{ij} \ge X_{ik} \wedge X_{jk} \quad \text{for all } 1 \le i, j, k \le n.$$

Under the three-point condition, the matrix $\mathbf{X}$ admits the interpretation as a rooted tree with leaves labeled in $[n]$ and distance $X_{ij}$ from the root to the point at which elements $i$ and $j$ branch apart in the tree. In this case, we regard $\mathbf{X}$ as a parameter in a phylogenetic model, not a medium in a physical sense. This choice of $\mathbf{X}$ hints at another potential application of permanental partition structures, which fits well with previous applications of partition models. We do not treat this case here.

4.3.2. *Mean-field permanental partition structures.* The $\alpha$-permanent of $\mathbf{H}[1,d;n]$, defined in (25), was computed in [10] by

$$\operatorname{per}_\alpha \mathbf{H}[1,d;n] = \sum_{k=1}^{n}\sum_{l=0}^{n} c(n,k,l)\alpha^k d^{n-l}, \tag{26}$$

where $c(n,k,l)$, called a *generalized rencontres number*, is the number of permutations of $[n]$ having exactly $k$ cycles and exactly $l$ fixed points. For $0 \leq l \leq k \leq n$, the generalized rencontres numbers satisfy the recursion

$$c(n,k,l) = c(n-1,k-1,l-1) + (n-l-1)c(n-1,k,l) + (l+1)c(n-1,k,l+1),$$

and, otherwise, $c(n,k,l) = 0$. Therefore, for $0 \leq d \leq 1$, we can express the $\textsc{Per}_n(\alpha,\beta,\mathbf{H}[1,d;n])$ distribution by

$$P_n^{\alpha,\beta,d}(\pi) = \beta^{\downarrow\#\pi}\frac{\prod_{b\in\pi}\sum_{k=1}^{\#b}\sum_{l=0}^{\#b} c(\#b,k,l)\alpha^k d^{\#b-l}}{\sum_{k=1}^{n}\sum_{l=0}^{n} c(n,k,l)(\alpha\beta)^k d^{n-l}}, \quad \pi\in\mathcal{P}_{[n]}\,.$$

**Proposition 4.7.** *For fixed $n \in \mathbb{N}$, $\alpha > 0$, $\beta \in [n]\cup[n,\infty)$ and $0 \leq d \leq 1$, the $\textsc{Per}_n(\alpha,\beta,\mathbf{H}[1,d;n])$ distribution on $\mathcal{P}_{[n]}$ is a canonical Gibbs ensemble with $v_n := \beta^{\downarrow n}$ and $w_n := \sum_{k=1}^{n}\sum_{l=0}^{n} c(n,k,l)\alpha^k d^{n-l}$.*

*Proof.* By Proposition 3.2 in [10], for $d \in [0,1]$ and $\alpha \in \mathbb{R}$, the $\alpha$-permanent of $\mathbf{H}[1,d;n]$ is given by (26). The correspondence between the $\textsc{Per}_n(\alpha,\beta,\mathbf{H}[1,d;n])$-law and the canonical $\mathrm{Gibbs}_{[n]}(v_\bullet, w_\bullet)$ ensemble with $v_\bullet$ and $w_\bullet$ as specified follows. □

Because the $\textsc{Per}_n(\alpha,\beta,\mathbf{H}[1,d;n])$-law has the canonical Gibbs form (11), it is exchangeable for every $0 \leq d \leq 1$. In particular, the $\textsc{Per}_n(\alpha,\beta,\mathbf{H}[1,1;n])$-law corresponds to the permanental partition structure in a uniformly homogeneous medium and the $\textsc{Per}_n(\alpha,\beta,\mathbf{H}[1,0;n])$-law corresponds to the permanental partition structure in a discrete medium.

**Theorem 4.8.** *The $\textsc{Per}_n(\alpha,\beta,\mathbf{X})$-law is exchangeable if and only if $\mathbf{X} = \mathbf{H}[1,d;n]$ for some $0 \leq d \leq 1$.*

*Proof.* One direction, that the $\textsc{Per}_n(\alpha,\beta,\mathbf{H}[1,d;n])$-law is exchangeable for all $d \in [0,1]$, follows from Proposition 4.7. In the other direction, suppose $P_n^{\alpha,\beta,\mathbf{X}}$ is exchangeable. In this case, for $n \geq 4$, consider the partition $\pi$ with blocks $\{1,2\},\{3\},\{4,\dots,n\}$, and let $\sigma \in \mathcal{S}_n$ be the permutation that transposes elements 2 and 3. Then $\pi^\sigma$ has blocks $\{1,3\},\{2\},\{4,\dots,n\}$ and, under exchangeability, must have the same probability of $\pi$. In this case, we have

$$P_n^{\alpha,\beta,\mathbf{X}}(\pi) = \beta^{\downarrow\#\pi}\frac{\prod_{b\in\pi}\operatorname{per}_\alpha \mathbf{X}[b]}{\operatorname{per}_{\alpha\beta}\mathbf{X}} = \beta^{\downarrow\#\pi^\sigma}\frac{\prod_{b\in\pi^\sigma}\operatorname{per}_\alpha \mathbf{X}[b]}{\operatorname{per}_{\alpha\beta}\mathbf{X}} = P_n(\pi^\sigma);$$

whence, $\operatorname{per}_\alpha \mathbf{X}[\{1,2\}] = \operatorname{per}_\alpha \mathbf{X}[\{1,3\}]$, from which we conclude $X_{12} = X_{13}$. Since the above argument holds for $1,2,3$ replaced by any distinct $i,j,k$, we must have $X_{ij} = X_{i'j'}$ for all $i \neq j$ and $i' \neq j'$. For $n \leq 3$, this result can be shown by omitting the last block $\{4,\dots,n\}$ in the above line of reasoning. This completes the proof. □

4.3.3. *Non-overlapping uniformly homogeneous subpopulations.* In this section, let $\mathbf{X} := (X_{ij},\ i,j \geq 1)$ be a 0-1 valued array representing a partition of $\mathbb{N}$. That is, for each $n \in \mathbb{N}$, the leading $n\times n$ submatrix $\mathbf{X}_n := (X_{ij},\ 1 \leq i,j \leq n)$ corresponds to some $\pi \in \mathcal{P}_{[n]}$ as in Section 2.1.

**Theorem 4.9.** *Let $\mathbf{X} := (X_{ij},\ i,j \geq 1)$ be an array representing a partition of $\mathbb{N}$ and let $\alpha > 0$, $k \geq 1$. For each $n \in \mathbb{N}$, let $\mathbf{X}_n := (X_{ij},\ 1 \leq i,j \leq n)$ be the leading $n\times n$ submatrix of $\mathbf{X}$. Then the collection $(P_n^{\alpha,k,\mathbf{X}_n},\ n \in \mathbb{N})$ of permanental partition distributions in (7) is consistent under subsampling.*

*Proof.* It is enough to establish that for $m \le n \in \mathbb{N}$, $\pi \in \mathcal{P}_{[m]}$ and $\pi^* \in \mathbf{D}^{-1}_{m,n}(\pi)$, the identity

$$\frac{\beta^{\downarrow \#\pi'} \operatorname{per}_\alpha(\pi \cdot \pi')}{\operatorname{per}_{\alpha\beta} \pi} = \frac{\sum_{\pi'' \in \mathbf{D}^{-1}_{m,n}(\pi')} \beta^{\downarrow \#\pi''} \operatorname{per}_\alpha(\pi^* \cdot \pi'')}{\operatorname{per}_{\alpha\beta} \pi^*} \tag{27}$$

holds for all $\pi' \in \mathcal{P}_{[m]}$. By induction, we need only show this for $m = n - 1$.

Fix $n \in \mathbb{N}$ and let $\pi \in \mathcal{P}_{[n]}$, $\pi^* \in \mathbf{D}^{-1}_{n,n+1}(\pi)$. Then, for any $\pi' \in \mathcal{P}_{[n]}$, each $\pi'' \in \mathbf{D}^{-1}_{n,n+1}(\pi')$ is obtained from $\pi'$ either by inserting element $n + 1$ into some $b \in \pi'$ or adding $\{n + 1\}$ as a singleton to $\pi'$. We write $b_* \in \pi$ to denote the block of $\pi$ to which $\{n + 1\}$ is added to obtain $\pi^*$. We have $\operatorname{per}_{\alpha\beta} \pi^* = (\alpha\beta + \#b_*) \operatorname{per}_{\alpha\beta} \pi$ and

$$\begin{aligned}
&\sum_{\pi'' \in \mathbf{D}^{-1}_{n,n+1}(\pi')} \beta^{\downarrow \#\pi''} \operatorname{per}_\alpha(\pi^* \cdot \pi'') = \\
&= \sum_{b' \in \pi'} \beta^{\downarrow \#\pi'} (\#(b_* \cap b') + \alpha) \operatorname{per}_\alpha(\pi \cdot \pi') + (\beta - \#\pi')\beta^{\downarrow \#\pi'} \alpha \operatorname{per}_\alpha(\pi \cdot \pi') \\
&= \beta^{\downarrow \#\pi'} \operatorname{per}_\alpha(\pi \cdot \pi')(\#b_* + \alpha\beta),
\end{aligned}$$

which completes the proof.

□

**Corollary 4.10.** *When $d = 0, 1$, the collection $(P_n^{\alpha,\beta,d},\ n \in \mathbb{N})$ of $\text{Per}_n(\alpha, \beta, \mathbf{H}[1, d; n])$ distributions is consistent.*

When $\mathbf{X}$ corresponds to a partition of $\mathbb{N}$, Kolmogorov's extension theorem and Theorem 4.9 imply the existence of a unique probability measure $P^{\alpha,k,\mathbf{X}}$ on $\mathcal{P}_\infty$ whose finite-dimensional distributions correspond to the $\text{Per}_n(\alpha, k, \mathbf{X}_n)$-law for each $n \in \mathbb{N}$. To our knowledge, in the literature on random partitions of $\mathbb{N}$, precise descriptions of probability distributions on $\mathbb{N}$ by consistent finite-dimensional marginals has been restricted to exchangeable and partially exchangeable random partition structures, see e.g. [32]. In Theorem 4.9, we encounter an explicit collection of finite-dimensional distributions that are not exchangeable, but nevertheless determine a unique probability measure on $\mathcal{P}_\infty$.

**Corollary 4.11.** *Let $(P_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ be a collection of permanental partition structures such that, for each $n \in \mathbb{N}$, $P_n^{\alpha,\beta,\mathbf{X}_n}$ is a probability distribution on $\mathcal{P}_{[n]}$. Then $(P_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ is both exchangeable and consistent if and only if $\beta \in \mathbb{N}$ and, for either $d = 0$ or $d = 1$, $\mathbf{X}_n := (X_{ij},\ 1 \le i, j \le n)$, where $(X_{ij},\ i, j \ge 1)$ is an infinite array with entries $X_{ij} = d$ if $i \ne j$ and $X_{ii} = 1$ for all $i \in \mathbb{N}$.*

*Proof.* Clearly, by Theorems 4.8 and 4.9, $\beta \in \mathbb{N}$ and $\mathbf{X} = \mathbf{H}[1, d; n]$, $0 \le d \le 1$, determines an exchangeable and consistent family of permanental partition structures. Conversely, suppose $(P_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ is exchangeable and consistent. Then, by Theorem 4.8, there is some $d \in [0, 1]$ such that $\mathbf{X}_n = \mathbf{H}[1, d; n]$ for all $n \in \mathbb{N}$. Furthermore, by the definition of the $\text{Per}_n(\alpha, \beta, \mathbf{X})$-partition structure, $\beta$ must be a positive integer. Under consistency, $(P_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ must satisfy

$$P_n^{\alpha,\beta,\mathbf{X}_n}(\pi) = \sum_{\pi' \in \mathbf{D}^{-1}_{n,n+1}(\pi)} P_{n+1}^{\alpha,\beta,\mathbf{X}_{n+1}}(\pi') \quad \text{for all } \pi \in \mathcal{P}_{[n]},$$

for each $n \in \mathbb{N}$. In the case of the permanental partition model (23) with $\beta = k \in \mathbb{N}$, consistency becomes

$$k^{\downarrow \#\pi} \frac{\mathrm{per}_{\alpha}(\mathbf{X}_n \cdot \pi)}{\mathrm{per}_{\alpha k} \mathbf{X}_n} = k^{\downarrow(\#\pi+1)} \alpha \frac{\mathrm{per}_{\alpha}(\mathbf{X}_n \cdot \pi)}{\mathrm{per}_{\alpha\beta} \mathbf{X}_{n+1}} + k^{\downarrow \#\pi} \frac{\mathrm{per}_{\alpha}(\mathbf{X}_n \cdot \pi)}{\mathrm{per}_{\alpha\beta} \mathbf{X}_{n+1}} \sum_{b \in \pi} \frac{\mathrm{per}_{\alpha}(\mathbf{X}_{n+1}[b \cup \{n+1\}])}{\mathrm{per}_{\alpha}(\mathbf{X}_n[b])};$$

whence,

$$\frac{\mathrm{per}_{\alpha\beta} \mathbf{X}_{n+1}}{\mathrm{per}_{\alpha\beta} \mathbf{X}_n} = (k - \#\pi)\alpha + \sum_{b \in \pi} \frac{\mathrm{per}_{\alpha}(\mathbf{X}_{n+1}[b \cup \{n+1\}])}{\mathrm{per}_{\alpha}(\mathbf{X}_n[b])}.$$

Since the left-hand side does not depend on $\pi$, we must have that the ratio $\mathrm{per}_{\alpha}(\mathbf{X}_{n+1}[b \cup \{n+1\}])/\mathrm{per}_{\alpha}(\mathbf{X}_n[b])$ is of the form $y + z\#b$, $z \geq 0$, for every $b \subseteq [n]$, in which case

$$\mathrm{per}_{\alpha}(\mathbf{X}_n[b]) = (y + z(\#b - 1)) \cdots (y + z)y \quad \text{for all } b \subseteq [n].$$

Clearly, $\mathrm{per}_{\alpha}(\mathbf{X}_n[[1]]) = \alpha$ forces $y = \alpha$. Furthermore, if $z > 0$, then

$$\mathrm{per}_{\alpha}(\mathbf{X}_n[[2]]) = \alpha^2 + d^2\alpha$$

implies $d = 1$. On the other hand, when $z = 0$, we have $\alpha^2 + d^2\alpha = \alpha^2$, which implies $d = 0$. □

4.3.4. *Similarity functions.* Rather than specifying a fixed similarity matrix $\mathbf{X}$, we could, instead, parameterize a permanental partition structure by a *similarity function* $\mathbf{K} : \mathcal{X} \times \mathcal{X} \to [0, 1]$, where $\mathcal{X}$ is assumed to be some abstract space in which particles are positioned, or suspended. Mimicking our definition of a similarity matrix, a similarity function $\mathbf{K}$ satisfies

- (i′) $\mathbf{K}(x, x) = 1$ for all $x \in \mathcal{X}$,
- (ii′) $0 \leq \mathbf{K}(x, x') \leq 1$ for all $x, x' \in \mathcal{X}$, and
- (iii′) $\mathbf{K}(x, x') = \mathbf{K}(x', x)$ for all $x, x' \in \mathcal{X}$.

This description gives a more vivid illustration of our prototypical phenomenon. We assume there are $n$ particles at positions $x_1, \ldots, x_n$ in space. The pairwise interactions among these particles are determined by the medium $\mathbf{K}$. Of course, for fixed $\mathbf{K}$ and $x_1, \ldots, x_n$, we can define a similarity matrix $\mathbf{X} := \mathbf{K}[x_1, \ldots, x_n] = (X_{ij}, 1 \leq i, j \leq n)$ with $X_{ij} := \mathbf{K}(x_i, x_j)$. Some benefits of this description are that it allows us to generate positions randomly, consider adding points, consider motion of particles in $\mathcal{X}$, etc. We discuss some generalization of this in Section 6, but leave its detailed study to future work.

## 5. Markovian Gibbs structures and the permanental partition process

We now discuss a Gibbs model for time-varying particle configurations, which we call *Markovian Gibbs structures*. We specifically consider Markov chain models in discrete-time. In general, a Markov chain on any finite space $\mathcal{Y}$ is determined by an initial distribution $\Upsilon_0$ and a transition probability measure $p(\cdot, \cdot)$, where $p(y, y')$ gives the probability of a transition from state $y$ to $y'$. In this section, we study the transition probabilities given in (8).

5.1. **Self-similar cut-and-paste chains.** In [8], we introduced a family of exchangeable and consistent Markov chains on the system $(\mathcal{P}_{[n]:k}, n \in \mathbb{N})$ of set partitions with at most $k$ blocks. Here we focus on a special sub-family of Ewens($\alpha$) cut-and-paste Markov chains, with $\alpha > 0$, which is reversible with respect to the Pitman-Ewens($-\alpha, k\alpha$)-laws:

$$P_n^{-\alpha,k\alpha}(\pi) = k^{\downarrow \#\pi} \frac{\prod_{b \in \pi} \alpha^{\uparrow \#b}}{(k\alpha)^{\uparrow n}}, \quad \pi \in \mathcal{P}_{[n]}.$$

The asymptotic behavior of these transition probabilities as both $\alpha$ and $k$ vary is summarized in the following proposition, whose proof is straightforward.

**Proposition 5.1.** *For $\alpha > 0$ and $k \geq 1$, let $p_n^{\alpha,k}$ be the finite-dimensional transition probabilities* (14). *The following asymptotic transition laws hold.*

- *For fixed $\alpha > 0$ and $k \to \infty$,* (14) *converges to*
$$p_n(\pi,\pi';\alpha) = \begin{cases} \alpha^{\#(\pi\wedge\pi')}\prod_{b\in\pi}\frac{\prod_{b'\in\pi'}\Gamma(\#(b\cap b'))}{\alpha^{\uparrow\#b}}, & \pi' \leq \pi \\ 0, & \text{otherwise}, \end{cases}$$
*the law of a Ewens($\alpha$)-fragmentation chain.*
- *For fixed $k \geq 1$ and $\alpha \to \infty$,* (14) *converges to*
$$p_n(\pi,\pi';k) = P_n(\pi';k) = k^{\downarrow\#\pi'}/k^n. \tag{28}$$
- *For fixed $k \geq 1$ and $\alpha \to 0$,* (14) *converges to*
$$p_n(\pi,\pi';k) = \begin{cases} k^{\downarrow\#\pi'}/k^{\#\pi}, & \pi \leq \pi' \\ 0, & \text{otherwise}, \end{cases}$$
*a discrete-time coalescent chain.*
- *For $\alpha/k \to 0$, $k \to \infty$ such that $\alpha \to \theta > 0$,* (14) *converges to*
$$p_n(\pi,\pi';\theta) = \begin{cases} \theta^{\#(\pi\wedge\pi')}\prod_{b\in\pi}\frac{\prod_{b'\in\pi'}(\Gamma(\#(b\cap b'))}{\theta^{\uparrow\#b}}, & \pi' \leq \pi \\ 0, & \text{otherwise}, \end{cases}$$
*the law of a Ewens($\theta$)-fragmentation chain.*
- *For $\alpha \to 0$ and $k \to \infty$,* (14) *converges to the unit mass:*
$$p_n(\pi,\pi') = \delta_\pi(\pi').$$

**Remark 5.2.** *Note that the asymptotic transition law as $\alpha \to \infty$ in* (28) *is independent of $\pi$. The weak limit of a sequence of partitions under $\alpha \to \infty$ is that of an independent and identically distributed sequence of partitions governed by the coupon-collector distribution on $\mathcal{P}_{[n]:k}$.*

**Remark 5.3.** *We call $\alpha$ the* diversity *parameter, as its magnitude relates to the dependence in the model: larger values of $\alpha$ correspond to more erratic behavior (the sequence is asymptotically i.i.d.), while small values of $\alpha$ lead to more controlled behavior, which tends toward local one-step transitions (pure coalescence).*

The transition probabilities in (14) resemble the form of a Gibbs measure (1) and motivates our consideration of *Markovian Gibbs structures* in the next section.

5.2. **Markovian Gibbs structures.** For $n \in \mathbb{N}$, let $H : 2^{[n]} \to \mathbb{R}$ be an *energy function* for the Markovian Gibbs transition probability
$$p_n(\pi,\pi') := \frac{a(\pi')}{Z_{a,H}(\pi)}\exp\left\{\sum_{b\in\pi\wedge\pi'} H(b)\right\}, \quad \pi,\pi' \in \mathcal{P}_{[n]}. \tag{29}$$
By putting $a(\pi') := k^{\downarrow\#\pi'}$ and $H(b) := \log(\alpha/k)^{\uparrow\#b}$, (29) reduces to the Ewens($\alpha$) cut-and-paste transition probability (14). We define a *canonical Markovian Gibbs structure* as any transition probability of the form
$$p_n(\pi,\pi';v_\bullet,w_\bullet) = \frac{v_{\#\pi'}}{B_n(v_\bullet,w_\bullet,\pi)}\prod_{b\in\pi\wedge\pi'} w_{\#b}, \quad \pi,\pi' \in \mathcal{P}_{[n]}, \tag{30}$$

for sequences of non-negative weights $v_\bullet := (v_n,\ n \in \mathbb{N})$ and $w_\bullet := (w_n,\ n \in \mathbb{N})$. A canonical Markovian Gibbs structure may also be called *mean-field* or *exchangeable*, as (30) satisfies the exchangeability criterion

$$p_n(\pi, \pi') = p_n(\pi^\sigma, \pi'^\sigma) \quad \text{for all } \sigma \in \mathcal{S}_n, \ \text{for all } \pi, \pi' \in \mathcal{P}_{[n]}\,.$$

Note that in the special case (14), the functions $a(\cdot)$ and $H(\cdot)$ depend on $\pi'$ and $b$, respectively, only through their cardinalities; hence, (14) is a canonical Markovian Gibbs structure with $v_\bullet := k^{\downarrow\bullet}$ and $w_\bullet := (\alpha/k)^{\uparrow\bullet}$.

**Corollary 5.4.** *For any $\alpha > 0$ and $k \geq 1$, the collection $(p_n^{\alpha,k},\ n \in \mathbb{N})$ of Ewens($\alpha$) cut-and-paste transition probabilities is a consistent collection of canonical Markovian Gibbs structures on $(\mathcal{P}_{[n]:k},\ n \in \mathbb{N})$.*

*Proof.* This proof is analogous to that of Corollary 4.11. □

Analogous to the characterization of canonical Gibbs ensembles by Pitman-Ewens($\alpha, \theta$) (Theorem 4.3), the Ewens($\alpha$) cut-and-paste chain is the unique consistent canonical Markovian Gibbs structure.

**Theorem 5.5.** *For non-negative sequences $v_\bullet$ and $w_\bullet$, let $(p_n(\cdot,\cdot; v_\bullet, w_\bullet),\ n \in \mathbb{N})$ be a collection of canonical Markovian Gibbs structures as in* (30)*. Then $(p_n(\cdot,\cdot; v_\bullet, w_\bullet),\ n \in \mathbb{N})$ is consistent if and only if $v_\bullet := k^{\downarrow\bullet}$ and $w_\bullet := \alpha^{\uparrow\bullet}$, for some $k \in \mathbb{N}$ and $\alpha > 0$.*

*Proof.* For convenience, we write (29) in the equivalent form

$$p_n(\pi, \pi') = \frac{a(\#\pi')}{Z(\pi)} \prod_{b \in \pi} \prod_{b' \in \pi'} w(\#(b \cap b')),$$

for $w(0) = 1$, where $w(n) := w_n$. Let $\pi, \pi' \in \mathcal{P}_{[n]}$, $\pi^* \in \mathbf{D}^{-1}_{n,n+1}(\pi)$ and write $w(\pi) = \prod_{b\in\pi} w_{\#b}$. The Gibbs form (29) and consistency (17) imply

$$\frac{a(\#\pi')}{Z(\pi)} w(\pi \wedge \pi') = \frac{a(\#\pi' + 1)}{Z(\pi^*)} w(\pi \wedge \pi') w(1) + \sum_{b \in \pi'} \frac{a(\#\pi')}{Z(\pi^*)} w(\pi^* \wedge \pi''_b),$$

where $\pi''_b$ is obtained by inserting element $n+1$ in block $b \in \pi'$; whence, for any $\pi'$ in the support of $p_n(\pi, \cdot)$,

$$C(\pi^*) := \frac{Z(\pi^*)}{Z(\pi)} = A(\#\pi') w(1) + \sum_{b' \in \pi'} W(\#(b' \cap b^*)),$$

where $W(i) := w(i+1)/w(i)$, $A(r) := a(r+1)/a(r)$ and $b^* \in \pi$ is the block into which $n+1$ is inserted to obtain $\pi^*$. The left side depends only on $\pi^*$, which implies $W(i+1) - W(i)$ is constant for all $i \geq 0$, and we may write $W(i+1) - W(i) = \beta$. For the moment, assume $W(i) = \alpha + \beta i$. We have

$$w(i) = W(i-1) W(i-2) \cdots W(0) = \prod_{j=0}^{i-1} (\alpha + \beta j) = \beta^i (\alpha/\beta)^{\uparrow i}.$$

There is no loss of generality in putting $\beta = 1$ so that $w(i) = \alpha^{\uparrow i}$, $i \geq 0$, giving

$$C(\pi^*) = A(r) w(1) + \sum_{b' \in \pi'} W(\#(b^* \cap b')) = \alpha A(r) + \alpha r + \#b^*, \quad r \leq \inf\{m \in \mathbb{N} : a(m+1) = 0\};$$

whence, $A(r) := \delta - r$ for some $\delta \in \mathbb{R}$. Without loss of generality, we assign $a(1) = \delta$ so that

$$a(r) = a(1)\prod_{j=1}^{r-1} A(j) = \delta \prod_{j=1}^{r-1}(\delta - j) = \delta^{\downarrow r}.$$

Since the Markovian Gibbs form (29) depends on $\pi'$ and $\pi \wedge \pi'$, but not $\pi$, both $\alpha$ and $\delta$ must be strictly positive. As a result, $A(r)$ will be negative for large enough $r$; and so we must have $\#\pi' < k$ for some integer $k < \infty$ and $a(k+1) = 0$ implies $\delta = k$.

In this case, the conditional Gibbs splitting rule is

$$p_n(\pi, \pi') = \frac{k^{\downarrow \#\pi'}}{Z(\pi)} \prod_{b\in\pi}\prod_{b'\in\pi'} \alpha^{\uparrow \#(b\cap b')}.$$

By (19), the normalizing constant $Z(\pi) = \mathrm{per}_{\alpha k}\, \pi$.

The following asymptotic cases arise.

- $k < \infty$ **and** $0 < \gamma < \infty$: In this case, (29) is exactly the Ewens($\gamma k$) cut-and-paste transition probability (14).
- $k < \infty$ **and** $\gamma = 0$: $\gamma = 0$ occurs if either $\beta = \infty$ or $\alpha = 0$ and, in this case, (29) is zero unless $\pi \wedge \pi' = \pi$ and the splitting rule has the form

$$p_n(\pi, \pi') = \frac{k^{\downarrow \#\pi'}}{k^{\#\pi}} I_{\{\pi' \wedge \pi = \pi\}},$$

  a discrete-time coalescent chain.
- $k < \infty$ **and** $\gamma = \infty$: $\gamma = \infty$ corresponds to $\beta = 0$, for which $w(i) = \alpha^i$ and $C(\pi^*) = \alpha A(r) + \alpha r'$; hence, $A(r') + r' = \delta > 0$ and $A(r') = \delta - r$ as above. In this case,

$$p_n(\pi, \pi') = \frac{k^{\downarrow \#\pi'}}{k^n}.$$

- $k = \infty$**,** $\gamma = 0$ **and** $0 < \gamma k = \theta < \infty$: We see that (29) becomes

$$p_n(\pi, \pi'; \theta) = \theta^{\#(\pi\wedge\pi')} \prod_{b\in\pi} \frac{\prod_{b'\in\pi'} \Gamma(\#(b\cap b'))}{\theta^{\uparrow \#b}} I_{\{\pi\wedge\pi' = \pi'\}}.$$

- $k = \infty$ **and** $0 < \gamma \le \infty$: This is the deterministic split into singletons, $p_n(\pi, \pi') = I_{\{\pi' = \mathbf{0}_n\}}$. Indeed, when $0 < \gamma \le \infty$ the limit of $p_n(\pi, \pi')$ in (29) as $k \to \infty$ is the degenerate distribution at $\{\#\pi' = n\} = \{\pi' = \mathbf{0}_n\}$. This is the limit of (14) as $k \to \infty$.
- $k = \infty$**,** $\gamma = 0$ **and** $\gamma k = 0$**; or** $k = \infty$**,** $\gamma = 0$ **and** $\gamma k = \infty$: In this case, $p_n(\pi, \pi') = I_{\{\pi = \pi'\}}$ is the degenerate distribution at $\pi$.

□

**Remark 5.6.** *In comparision to Theorem 4.3, the characterization in Theorem 5.5 only permits the case corresponding to $\alpha < 0$ and $\theta = -k\alpha$ for $k \in \mathbb{N}$; the canonical Markovian Gibbs ensemble cannot be supported on all of $\mathcal{P}_{[n]}$, for all $n \in \mathbb{N}$. The difference between the cases arises because, when inserting a new element $n+1$ into partitions $(\pi, \pi')$, a new block can form in $\pi \wedge \pi'$ without forming in either $\pi$ or $\pi'$.*

5.3. **Permanental Markovian Gibbs structures.** For a certain choice of $\alpha,\beta\in\mathbb{R}$ and $\mathbf{X}\in\mathbb{R}^{n\times n}$, we defined the permanental transition probability on $\mathcal{P}_{[n]}$ in (8). The transition probability $p_n^{\alpha,\beta,\mathbf{X}}$ in (8) has the form of a Markovian Gibbs structure (29) with $a(\pi'):=\beta^{\downarrow\#\pi'}$, $H(b):=\log\mathrm{per}_\alpha\mathbf{X}[b]$ and $Z_{a,H}(\pi):=\mathrm{per}_{\alpha\beta}(\mathbf{X}\cdot\pi)$. An important special case of (8) arises when $\alpha>0$, $\beta=k\in\mathbb{N}$ and $\mathbf{X}=\mathbf{1}_n$, the $n\times n$ matrix of all ones, in which case (8) coincides with the Ewens($\alpha$) Markovian Gibbs structure (14).

**Proposition 5.7.** *Let $\alpha>0$, $\beta\in[n]\cup[n,\infty)$ and $\mathbf{X}$ be a similarity matrix, then, for every $\pi\in\mathcal{P}_{[n]}$, $p_n(\pi,\cdot)$ in* (8) *is a probability distribution on $\mathcal{P}_{[n]}$. It follows that $p_n^{\alpha,\beta,\mathbf{X}}$ in* (8) *is a transition probability on $\mathcal{P}_{[n]}$.*

*Proof.* This follows from Proposition 4.4 since, for any similarity matrix $\mathbf{X}$, the Hadamard product $\mathbf{X}\cdot\pi$ is also a similarity matrix, for every $\pi\in\mathcal{P}_{[n]}$. □

**Definition 5.8** (Permanental Markovian Gibbs structures)**.** *For $n\in\mathbb{N}$, $\alpha>0$, $\beta\in[n]\cup[n,\infty)$ and any similarity matrix $\mathbf{X}$, we call $p_n^{\alpha,\beta,\mathbf{X}}$ in* (8) *the* $\mathrm{Per}_n(\alpha,\beta,\mathbf{X})$*-Markovian Gibbs structure with* diversity $\alpha$, mixing coefficient $\beta$ *and* similarity matrix $\mathbf{X}$. *Alternatively, we may refer to $p_n^{\alpha,\beta,\mathbf{X}}$ as the* permanental Markovian Gibbs structure, *or* permanental transition probability, *with parameter $(\alpha,\beta,\mathbf{X})$.*

**Theorem 5.9.** *The $\mathrm{Per}_n(\alpha,\beta,\mathbf{X})$-transition measure is reversible with respect to the $\mathrm{Per}_n(\alpha\beta,\beta,\mathbf{X})$-distribution.*

*Proof.* For $\alpha>0$ and $\beta\in[n]\cup[n,\infty)$, $\alpha\beta>0$, and so, by Proposition 4.4, $(\alpha\beta,\beta,\mathbf{X})$ is in the parameter space of the permanental partition distribution. It is immediately clear that $p_n^{\alpha,\beta,\mathbf{X}}$ in (8) satisfies the detailed-balance condition (18) with respect to $P_n^{\alpha\beta,\beta,\mathbf{X}}$ in (7). □

**Theorem 5.10.** *The permanental transition probability in* (8) *is a canonical Markovian Gibbs structure* (30) *if and only if $\mathbf{X}$ is homogeneously symmetric. In this case, $p_n^{\alpha,k,\mathbf{X}}$ satisfies* (30) *for $v_\bullet:=k^{\downarrow\bullet}$ and $w_\bullet:=\sum_{k=1}^{\bullet}\sum_{l=0}^{\bullet}c(\bullet,k,l)\alpha^k d^{\bullet-l}$, for some $d\in[0,1]$.*

*Proof.* This proof is analogous to that of Theorem 4.8. □

5.3.1. *The case $\beta\in\mathbb{N}$.* We now specialize to the case $\beta\in\mathbb{N}$ which, by Definition 5.8, is in the parameter space of the $\mathrm{Per}_n(\alpha,\beta,\mathbf{X})$-transition probability for every $n\in\mathbb{N}$. For $k\geq 1$, the $\mathrm{Per}_n(\alpha,k,\mathbf{X})$-transition probability in (8) is a transition probability on the subspace $\mathcal{P}_{[n]:k}$.

**Corollary 5.11.** *For every $n\in\mathbb{N}$, the Ewens($\alpha$) cut-and-paste transition probability in* (14) *coincides with the $\mathrm{Per}_n(\alpha/k,k,\mathbf{1}_n)$-transition probability in* (8).

Another special case arises when $\mathbf{X}=I_n$, the $n\times n$ identity matrix. In this case, (8) simplifies to

$$p_n^{\alpha,k,I_n}(\pi,\pi'):=k^{\downarrow\#\pi'}/k^n,\quad \pi,\pi'\in\mathcal{P}_{[n]:k},$$

which is independent of $\pi$ and $\alpha$. Hence, when $\mathbf{X}=I_n$, the permanental transition probabilities govern an i.i.d. sequence of coupon collector partitions on $k$ coupons.

We also point out that, in the cases $\mathbf{X}=\mathbf{1}_n$ and $I_n$, the $\mathrm{Per}_n(\alpha,k,\mathbf{X})$-transition probability is consistent. We discuss consistent collections of transition probabilities in the next section.

5.4. **Consistent Markovian Gibbs structures.** Recall, from Section 2.4, that a family of transition probabilities $(p_n,\ n \in \mathbb{N})$ on $(\mathcal{P}_{[n]},\ n \in \mathbb{N})$ is consistent if and only if, for every $\pi, \pi' \in \mathcal{P}_{[m]}$,

$$p_m(\pi, \pi') = p_n(\pi^*, \mathbf{D}_{m,n}^{-1}(\pi')) \quad \text{for all } \pi^* \in \mathbf{D}_{m,n}^{-1}(\pi).$$

**Theorem 5.12.** *Let $\mathbf{X}$ be an infinite array corresponding to a partition of $\mathbb{N}$, and, for each $n \in \mathbb{N}$, let $\mathbf{X}_n := (X_{ij},\ 1 \le i, j \le n)$ be its leading $n \times n$ submatrix. If, for each $n \in \mathbb{N}$, $p_n^{\alpha,k,\mathbf{X}_n}$ denotes the $\mathrm{Per}_n(\alpha, k, \mathbf{X}_n)$-transition probability on $\mathcal{P}_{[n]:k}$, then $(p_n^{\alpha,k,\mathbf{X}_n},\ n \in \mathbb{N})$ is a consistent collection of Markovian Gibbs structures.*

*Proof.* This proof is analogous to that of Theorem 4.9. □

By Theorem 5.12, we can construct a Markov chain on $\mathcal{P}_{[\infty]:k}$, partitions of $\mathbb{N}$ with at most $k$ blocks, that is *not* exchangeable. By combining Theorem 5.10 and 5.12, we arrive at a characterization of the class of exchangeable and consistent permanental transition probabilities.

**Corollary 5.13.** *Let $(p_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ be a family of permanental Markovian Gibbs structures such that, for each $n \in \mathbb{N}$, $p_n^{\alpha,\beta,\mathbf{X}_n}$ is a transition probability on $\mathcal{P}_{[n]}$. Then $(p_n^{\alpha,\beta,\mathbf{X}_n},\ n \in \mathbb{N})$ is exchangeable and consistent if and only if $\beta \in \mathbb{N}$ and, for either $d = 0$ or $d = 1$, $\mathbf{X}_n := (X_{ij},\ 1 \le i, j \le n)$, where $(X_{ij},\ i, j \ge 1)$ is an infinite array with entries $X_{ij} = d$ if $i \ne j$ and $X_{ii} = 1$ for all $i \in \mathbb{N}$.*

## 6. Equivariance, exchangeability and lack of interference

In this section, we discuss both the Permanental partition structures and Permanental Markovian Gibbs structures simultaneously by referring generically to the *permanental model*.

6.1. **Equivariance.** In a homogeneous enviroment, the permanental model is exchangeable; however, in any other environment, it is not. In other situations, the lack of homogeneity implies that the positions of the particles affect pairwise interactions and, hence, the law governing the configuration. In a non-exchangeable model in which particles have fixed positions in space (or, more generically, fixed covariates), the natural model invariance is with respect to the action of the symmetric group on the similarity matrix parameter, which reflects a relabeling of the positions $x_1, \ldots, x_n$, but not any substantive change to the configuration of the particles in space (as we are assuming particles are physically identical). Such a model is called *equivariant* with respect to the symmetric group. Note that exchangeability is also defined as an invariance with respect to the symmetric group; however, exchangeability is a property of a *distribution*, whereas equivariance is a property of the *family*. In particular, a probability distribution $P_n$ on $\mathcal{P}_{[n]}$ is exchangeable if, for each $\pi \in \mathcal{P}_{[n]}$, $P_n(\pi) = P_n(\pi^\sigma)$, for all $\sigma \in \mathcal{S}_n$. On the other hand, a family $(P_n^{\mathbf{X}})$ indexed by a similarity matrix $\mathbf{X}$ is equivariant with respect to the symmetric group if, for every permutation $\sigma : [n] \to [n]$, $(P_n^{\mathbf{X}^\sigma})$ determines the same family as $(P_n^{\mathbf{X}})$, where $\mathbf{X}^\sigma := (X_{\sigma(i)\sigma(j)},\ 1 \le i, j \le n)$. Equivariance holds, for instance, if, for each $\pi \in \mathcal{P}_{[n]}$, $P_n^{\mathbf{X}}(\pi) = P_n^{\mathbf{X}^\sigma}(\pi^\sigma)$ for all $\sigma \in \mathcal{S}_n$. By a symmetry property of the $\alpha$-permanent, the permanental model is equivariant with respect to the symmetric group acting on $[n]$, as we now show.

**Proposition 6.1.** *For every $n \in \mathbb{N}$, both the $\mathrm{Per}_n(\alpha, \beta, \mathbf{X})$-partition structure and Markovian Gibbs structure are equivariant with respect to the symmetric group.*

*Proof.* For any matrices $X, Y$ and $\sigma \in \mathcal{S}_n$, $\operatorname{per}_\alpha X = \operatorname{per}_\alpha(X^\sigma)$ and $(X^\sigma) \cdot (Y^\sigma) = (X \cdot Y)^\sigma$. Hence, for any $\pi \in \mathcal{P}_{[n]}$ and $(\alpha, \beta, \mathbf{X})$ in the parameter space of the permanental partition model,

$$P_n^{\alpha,\beta,\mathbf{X}}(\pi) = \beta^{\downarrow\#\pi} \frac{\operatorname{per}_\alpha(\mathbf{X} \cdot \pi)}{\operatorname{per}_{\alpha\beta} \mathbf{X}} = \beta^{\#\pi^\sigma} \frac{\operatorname{per}_\alpha(\mathbf{X}^\sigma \cdot \pi^\sigma)}{\operatorname{per}_{\alpha\beta} \mathbf{X}^\sigma} = P_n^{\alpha,\beta,\mathbf{X}^\sigma}(\pi^\sigma),$$

for all $\sigma \in \mathcal{S}_n$. The same calculation shows that the $\mathrm{Per}_n(\alpha, \beta, \mathbf{X})$-family of transition measures is equivariant. □

Equivariance is important as it ensures that the law of $\Pi \sim P_n^{\mathbf{X}}$ depends on $\mathbf{X}$ only through its fundamental structure (i.e. the pairwise interactions/similarities among particles) and not any superfluous variates, e.g. the labels $1, \ldots, n$ assigned the particles.

6.2. **Generating exchangeable Gibbs structures by mixing.** Although the $\mathrm{Per}_n(\alpha, \beta, \mathbf{X})$ model is not exchangeable in general, we can introduce $\mathbf{X}$ as an exchangeable *random environment* governed by a distribution $\Sigma$ on the space of similarity matrices. In this case, the unconditional $\mathrm{Per}_n(\alpha, \beta; \Sigma)$-model (defined below) gives a Bayesian model for particle configurations. In particular, assume $\Sigma$ is a *weakly exchangeable* probability measure, i.e. $\mathbf{X} \sim \Sigma$ satisfies $\mathbf{X}^\sigma =_{\mathcal{L}} \mathbf{X}$ for every $\sigma \in \mathcal{S}_n$. In this case, we define the unconditional permanental models by

$$P_n^{\alpha,\beta;\Sigma}(\pi) := \int P_n^{\alpha,\beta,\mathbf{X}}(\pi)\Sigma(d\mathbf{X}) \qquad \text{and}$$
$$p_n^{\alpha,\beta;\Sigma}(\pi, \pi') := \int p_n^{\alpha,\beta,\mathbf{X}}(\pi, \pi')\Sigma(d\mathbf{X}).$$

**Proposition 6.2.** *Let $\Sigma$ be a weakly exchangeable probability measure on $n \times n$ similarity matrices. Then the unconditional $\mathrm{Per}_n(\alpha, \beta; \Sigma)$-partition structure $P_n^{\alpha,\beta;\Sigma}$ and $\mathrm{Per}_n(\alpha, \beta; \Sigma)$-Markovian Gibbs structure $p_n^{\alpha,\beta;\Sigma}$ are exchangeable.*

*Proof.* This is a straightforward consequence of weak exchangeability of $\Sigma$ and Proposition 6.1.

□

**Theorem 6.3.** *Let $\Sigma$ be an exchangeable probability measure on $\mathcal{P}_\infty$. Then the permanental model with parameter $(\alpha, \beta; \Sigma)$ is both exchangeable and consistent and, therefore, determines a unique process on $\mathcal{P}_\infty$.*

*Proof.* This follows from Theorems 4.9 and 5.12 and Proposition 6.2. □

The results in this section hint at probability models for particle configurations in a random enviroment. This relates to some previous literature on random walks in random environments (RWRE), see e.g. [20]. Based on the developments in Sections 4 and 5, we have a description of a RWRE with explicit marginal distributions, which is a topic for future study.

6.3. **Consistency under subsampling and lack of interference.** We have shown (Theorems 4.9 and 5.12) that, under certain choices of the similarity matrix $\mathbf{X}$, the permanental model is consistent under subsampling. In the context of physical models, consistency under subsampling reflects an inherent *lack of interference* among the particles in the system. That is, given a configuration $x_1, \ldots, x_n, x_{n+1}$ of $n + 1$ particles in space, the law governing the clustering of the particles at positions $x_1, \ldots, x_n$, ignoring the particle at position $x_{n+1}$, is the same as the law of a system having only $n$ particles at positions $x_1, \ldots, x_n$. In other words, an additional particle, or particles, does not interfere with the law governing the configuration of the rest of the system. In general, the permanental partition models do not exhibit lack of interference, just as many physical systems do not possess such a property. In many physical systems, addition, or removal, of a particle perturbs how the rest of the system interacts in complicated ways; therefore, although a lack of consistency is detrimental from a statistical inference perspective, see e.g. [27], lack of interference exhibits a flexibility, rather than a defect, of the permanental partition model in modeling physical systems.

6.4. **Permanental Gibbs structures on $k$-colorings.** For fixed $k \geq 1$, the permanental distribution and transition measures can be defined on the space $[k]^{[n]}$ of $[k]$-valued sequences of length $n$, called *$k$-colorings* of $[n]$. In addition to (19), the $\alpha$-permanent satisfies the following more general identity with respect to $k$-colorings (Lemma 3 in [29]). Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and write $\alpha_\bullet := \alpha_1 + \cdots + \alpha_k$. Then, for any $n \times n$ matrix $\mathbf{X}$,

$$\mathrm{per}_{\alpha_\bullet} \mathbf{X} = \sum_{(w_1,\ldots,w_k)} \mathrm{per}_{\alpha_1}(\mathbf{X}[w_1]) \cdots \mathrm{per}_{\alpha_k}(\mathbf{X}[w_k]), \tag{31}$$

where the sum is over ordered collections $(w_1, \ldots, w_k)$ of disjoint subsets of $[n]$ such that $\bigcup_{i=1}^k w_i = [n]$. Note that, for $k = \beta$ and $\alpha_1 = \cdots = \alpha_k = \alpha$, (31) reduces to (19). In this way, (31) is a refinement of (19) in the special case where $\beta$ is a non-negative integer. The identity (31) suggests a probability distribution on $k$-colorings of $[n]$ by

$$P_n^{\alpha,\mathbf{X}}(x) := \frac{\prod_{j=1}^k \mathrm{per}_{\alpha_j} \mathbf{X}[x_j]}{\mathrm{per}_{\alpha_\bullet} \mathbf{X}}, \quad x \in [k]^{[n]}, \tag{32}$$

where $\alpha := (\alpha_1, \ldots, \alpha_k)$ and $x_j := \{i \in [n] : x^i = j\}$ is the set of indices with color $j$ in $x$. When $\mathbf{X} = \mathbf{1}_n$, (32) corresponds to the Dirichlet-Multinomial distribution on $[k]^{[n]}$ with parameter $(\alpha_1, \ldots, \alpha_k)$.

For a $k \times k$ matrix $\alpha := (\alpha_{ij}, 1 \leq i, j \leq k)$, we define the permanental Markovian Gibbs structure on $[k]^{[n]}$ by

$$p_n^{\alpha,\mathbf{X}}(x, x') := \prod_{i=1}^k \frac{\prod_{j=1}^k \mathrm{per}_{\alpha_{ij}} \mathbf{X}[x_i \cap x'_j]}{\mathrm{per}_{\alpha_{i\bullet}} \mathbf{X}[x_i]}, \quad x, x' \in [k]^{[n]}, \tag{33}$$

where $\alpha_{i\bullet} := \sum_{j=1}^k \alpha_{ij}$ for each $i = 1, \ldots, k$. Note that (33) is reversible with respect to $P_n^{\alpha',\mathbf{X}}$ in (32) with $\alpha' := (\alpha_{1\bullet}, \ldots, \alpha_{k\bullet})$. Using the same arguments, one can obtain analogous outcomes for exchangeable, consistent and reversible permanental Gibbs structures and Markovian Gibbs structures on $[k]^{[n]}$.

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