Comments on: "Big Data" Asymptotics via Approximate Message Passing

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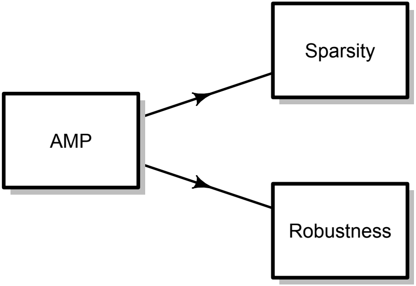
Kill two birds with one stone:



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- Extra Gaussian noise:

$$\widetilde{F} = F * N\big(0, \tau_\infty^2\big)$$

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- How much of is the extra noise intrinsic?
  - The smallest τ_∞^2 in HD M-estimation (Bean et al, 13)
  - HD Hájek-Le Cam-type convolution theorem for location-scale invariant $\widehat{\theta}$ and/or dense θ?

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when $\|\mathbf{Z}^\top \mathbf{X} - \mathbf{I}_{p\times p}\|_\infty$ is small, where $\mathbf{\Sigma} = \mathbb{E}\,\mathbf{X}^\top \mathbf{X}$

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$$\begin{aligned} \frac{1}{p}\sum_{j=1}^p \mathbb{E}\left(t(\widehat{\theta}_j^M) - \theta_j\right)^2 &\approx \mathrm{BayesRisk}\left(t(\cdot), V(\widetilde{\Psi}, \widetilde{F}), G_n\right) \\ &= \int \mathbb{E}\left(t(\theta + \widetilde{\varepsilon}) - \theta\right)^2 G_n(d\theta) \end{aligned}$$

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  - General $t(\cdot)$ (Robbins, 56); Linear $t(\cdot)$ (James-Stein, 61; Efron-Morris, 72)